1. Consider the following matrices:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}, \qquad A^TA = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}, \qquad AA^T = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix}.$$

   (a) Find the eigenvalues σ_1^2, σ_2^2 and unit eigenvectors v_1, v_2 of A^TA.
   (b) For the $\sigma_i \neq 0$, compute $u_i = Av_i/\sigma_i$ and verify that indeed $||u_i|| = 1$. Find the other u_i by computing the other unit eigenvector of AA^T.
   (c) Construct the left polar decomposition, $A = UP$.
   (d) Construct the singular value decomposition (SVD), $A = U\Sigma V^T$.
   (e) Write down orthonormal bases for each the "four fundamental subspaces": the row space R_A, the nullspace N_A, the column space R_{A^T}, and the left nullspace N_{A^T}.
   (f) Describe *all* matrices that have the same four fundamental subspaces.
   (g) Find a left, right, and pseudoinverse of A, or explain why it doesn't exist.

2. Compute the polar and singular value decomposition of the rotation matrix

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

   where $a, b \in \mathbb{R}$.

3. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.
   (a) Construct the singular value decomposition of A.
   (b) Write down orthonormal bases for each the "four fundamental subspaces": the row space R_A, the nullspace N_A, the column space R_{A^T}, and the left nullspace N_{A^T}.
   (c) Find a left inverse, right inverse, and pseudoinverse of A, or explain why it doesn't exist.

4. Let $f\colon \mathbb{R}^4 \to \mathbb{R}^3$ be the linear map defined by $f(x) = Mx$ for $x \in \mathbb{R}^4$ where $M = ABC$ and

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{-1}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

   (a) Define the adjoint map $f^*\colon \mathbb{R}^3 \to \mathbb{R}^4$ (under the standard Euclidean inner product) and express it in terms of M.
   (b) Find a singular value decomposition (SVD) of M. (*Hint*: Observe that A^TA and C^TC are diagonal.)
   (c) Find all $x \in \mathbb{R}^4$ with $\|x\| = 1$ so that $\|Mx\|$ is maximized.
   (d) Describe the eigenvalues and eigenvectors of M^TM.
   (e) Find the least square solution for $Mx = b$ with $\|x\|_2$ minimal where $b = (1, 1, 1)^T$. (*Hint*: Use the pseudo-inverse of M.)