Lecture 1.4: Quotient spaces

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Overview

In the previous lecture, we saw several ways to "multiply" vector spaces together, and we constructed the:

- complement of a subspace
- direct sum of two subspaces
- direct product of two vector spaces.

In this lecture, we will see how to divide a vector space by a subspace.

This is called a quotient space. It can be thought of as the analogue of modular arithmetic for vector spaces.

We will also use this to compute the dimension of the sum of two subspaces.

Congruence of subspaces

Definition

If Y is a subspace of X, then two vectors $x_1, x_2 \in X$ are congruent modulo Y, denoted $x_1 \equiv x_2 \pmod{Y}$, if $x_1 - x_2 \in Y$.

Proposition (exercise)

Congruence modulo Y is an equivalence relation, i.e., it is:

(i) **reflexive**: $x \equiv x$ for all $x \in X$;

(ii) **symmetric**: $x \equiv y$ implies $y \equiv x$;

(iii) **transitive**: $x \equiv y$ and $y \equiv z$ implies $x \equiv z$. □

The equivalence classes are called congruence classes mod Y, or cosets. Denote the class containing x by $\{x\}$. [Sometimes written $\overline{x}$ or $x + Y := \{x + y \mid y \in Y\}$.]

Example

Let $X = \mathbb{R}^3$, $\quad Y = \{(x, y, 0) \mid x, y \in \mathbb{R}\} =$ xy-plane, $\quad Z = \{(0, 0, z) \mid z \in \mathbb{R}\} =$ z-axis.

- $v \equiv w$ mod Y iff they lie on the same horizontal plane.
- $v \equiv w$ mod Z iff they lie on the same vertical line.

Quotient spaces

Let X/Y denote the set of equivalence classes in X, modulo Y.

This can be made into a vector space by defining addition and scalar multiplication as follows:

$$\{x\} + \{z\} := \{x + z\}, \qquad a\{x\} := \{ax\}.$$

We need to check that this is well-defined, i.e., that it is *independent of the choice of representative* from the classes.

This means showing (HW exercise) that if $x_1 \equiv x_2 \mod Y$ and $z_1 \equiv z_2 \mod Y$, then

$$\{x_1\} + \{z_1\} = \{x_2\} + \{z_2\}, \qquad a\{x_1\} = a\{x_2\}.$$

Definition

The vector space X/Y is called the quotient space of X modulo Y.

Alternate notations

Since $\{x\}$ is sometimes written $\overline{x}$, or $x + Y := \{x + y \mid y \in Y\}$, then addition and multiplication becomes:

- $\overline{x} + \overline{z} = \overline{x + z}$, and $a\overline{x} = \overline{ax}$;
- $(x + Y) + (z + Y) = x + z + Y$, and $a(x + Y) = ax + Y$.

Dimension of quotient spaces

Theorem 1.6

If Y is a subspace of a finite-dimensional vector space X, then $\dim Y + \dim X/Y = \dim X$.

Proof

Corollary

If a subspace Y of a finite-dimensional space X has $\dim Y = \dim X$, then $Y = X$. □

Dimension of sums

Theorem 1.7

Let U, V be subspaces of a finite-dimensional space X with $U + V = X$. Then

$$\dim X = \dim U + \dim V - \dim(U \cap V).$$

Proof