Lecture 1.5: Dual vector spaces

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Scalar functions

Let X be a vector space over a field K. A scalar function is any function from X to K.

A scalar function $\ell\colon X \to K$ is linear if

- $\ell(x+y) = \ell(x) + \ell(y)$, for all $x, y \in X$;
- $\ell(cx) = c\ell(x)$, for all $x \in X$, $c \in K$.

Or equivalently, if

$$\ell(c_1x_1 + \cdots + c_nx_n) = c_1\ell(x_1) + \cdots + c_n\ell(x_n), \qquad \text{for all } c_i \in K,\ x_i \in X\,.$$

Definition

The set of linear scalar functions $\ell\colon X \to K$ is a vector space called the dual of X, and denoted X'.

Addition and scalar multiplication is defined naturally:

- Addition: $(\ell + m)(x) := \ell(x) + m(x)$,
- Scalar multiplication: $(c\ell)(x) := c\ell(x)$.

Examples of scalar functions

Example 1

Let $X = \mathcal{C}([0,1],\mathbb{R})$, the continuous functions $[0,1] \to \mathbb{R}$, and fix $t_1, \dots, t_n \in [0,1]$. The following are linear scalar functions:

- $\ell(f) = f(t_1)$;
- $\ell(f) = \sum_{i=1}^{n} a_i f(t_i), \quad a_i \in \mathbb{R}$;
- $\ell(f) = \int_0^1 f(t)\,dt$.

Example 2

Let $X = \mathcal{C}^\infty(\mathbb{R})$ be the set of smooth functions $\mathbb{R} \to \mathbb{R}$. For a fixed $t_0 \in \mathbb{R}$,

$$\ell := \sum_{i=1}^{n} a_i \left.\frac{d^i}{dt^i}\right|_{t=t_0}, \qquad \ell\colon f \longmapsto \sum_{i=1}^{n} a_i \left.\frac{d^i f}{dt^i}\right|_{t=t_0}$$

is a linear scalar function (i.e., an element of X').

The dual space

If $\dim X = n$, then $X \cong K^n$. Thus, we can associate a vector $x \in X$ with an n-tuple $x = (c_1, \dots, c_n)$ of scalars.

For any fixed $a_1, \dots, a_n \in K$, the function

$$\ell \colon X \longrightarrow K, \qquad \ell(x) = a_1 c_1 + \cdots + a_n c_n \tag{1}$$

is linear, i.e., $\ell \in X'$.

Theorem 1.8

If $\dim X = n < \infty$, then every $\ell \in X'$ can be written as in Eq. (1).

Proof

The dual space

Corollary 1.9

If $\dim X < \infty$, then $X \cong X'$.

One way to think of this is to:

1. associate a vector $x \in X$ with a column vector,
2. associate a scalar function $\ell \in X'$ with a row vector.

Notation

A linear function $\ell \in X'$ applied to a vector $x \in X$ depends on the n-tuples $(c_1, \ldots, c_n)$ for x and $(a_1, \ldots, a_n)$ for ℓ. We can use scalar product notation

$$(\ell, x) := \ell(x).$$

Sometimes, elements $\ell \in X'$ are called co-vectors, or dual vectors.

Definition

Let $x_1, \ldots, x_n$ be a basis for X. The dual basis in X' is $\ell_1, \ldots, \ell_n$, where

$$(\ell_i, x_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Think of ℓ_i as the function that "picks off" the coefficient of x_i.

Duality in infinite dimensional spaces

Consider the vector space

$$X = \ell^1(\mathbb{R}) := \Big\{(x_1, x_2, \dots) \mid x_i \in \mathbb{R},\ \sum_{i=1}^{\infty} |x_i| < \infty\Big\}.$$

Given vectors $y = (a_1, a_2, \dots)$ and $x = (c_1, c_2, \dots)$,

$$(y, x) = \sum_{i=1}^{\infty} a_i c_i < \infty,$$

so every $y \in X$ defines a co-vector in X'.

But there are others! If $z = (1, 1, 1, \dots)$,

$$(z, x) = \sum_{i=1}^{\infty} c_i < \infty,$$

but $z \notin X$.

The double dual

The scalar product (ℓ, x) is a bilinear function of ℓ and x. That is, if we fix one argument, it is linear in the other. Equivalently,

$$\underbrace{(a\ell, x)}_{=a\ell(x)} = a(\ell, x) = \underbrace{(\ell, ax)}_{\ell(ax)} \qquad \text{for all } x \in X,\ \ell \in X',\ a \in K.$$

If $\dim X = n < \infty$, then every linear scalar function $X \to K$ is of the form

$$(\ell, x), \qquad \text{for some fixed } \ell = (a_1, \dots, a_n) \in K^n.$$

Since X' is a vector space, it has a dual, called the double dual of X, and denoted $X'' := (X')'$. Every linear scalar function $X' \to K$ is of the form

$$(\ell, x), \qquad \text{for some fixed } x = (c_1, \dots, c_n) \in K^n.$$

Key points

Let $x_1, \dots, x_n$ be a basis of X

- Think of the dual basis $\ell_1, \dots, \ell_n$ as "*pick-off functions*"
- Think of elements in the double dual as "*evaluation functions*"

The bilinear function (ℓ, x) naturally identifies X'' with X.