Lecture 2.1: Rank and nullity

Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

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Preliminaries

Goal

Abstract the concept of a matrix as a linear mapping between vector spaces.

Advantages:

- simple, transparent proofs;
- better handles infinite dimensional spaces.

Definition (revisted)

A linear map (or *mapping*, *transformation*, or *operator*) between vector spaces X and U over K is a function $T\colon X \to U$ that is:

(i) additive: $T(x+y) = T(x) + T(y)$, for all $x, y \in X$,

(ii) homogeneous: $T(ax) = aT(x)$, for all $x \in X$, $a \in K$.

The domain space is X and the target space is U.

Usually we'll write Tx for $T(x)$, and so additivity is just the distributive law:

$$T(x+y) = Tx + Ty.$$

Examples of linear maps

(i) Any isomorphism;

(ii) $X = U = \{\text{polynomials of degree } < n \text{ in t}\}, \quad T = \frac{d}{dt}$.

(iii) $X = U = \mathbb{R}^2, \quad T =$ rotation about the origin.

(iv) X any vector space, $U = K$ (1-dimensional), T any $\ell \in X'$.

(v) $X = U = \mathcal{C}([0,1], \mathbb{R})$, $g \in X$. $\quad (Tf)(x) = \displaystyle\int_0^1 f(y)g(x-y)\,dy.$

(vi) $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, $u = Tx$, where $u_i = \displaystyle\sum_{j=1}^{n} t_{ij}x_j, \quad i = 1, \dots, m.$

(vii) $X = U = \{\text{piecewise cont. } [0,\infty) \to \mathbb{R} \text{ of ``exponential order''}\}$,
$(Tf)(s) = \displaystyle\int_0^\infty f(t)e^{-st}\,dt.$ "*Laplace transform*"

(viii) $X = U = \{\text{functions with } \int_{-\infty}^{\infty} |f(x)|\,dx < \infty\}$,
$(Tf)(\xi) = \displaystyle\int_{-\infty}^{\infty} f(x)e^{i\xi x}\,dx.$ "*Fourier transform*"

Basic properties

Theorem 2.1

Let $T\colon X \to U$ be a linear map.

(a) The image of a subspace of X is a subspace of U.

(b) The preimage of a subspace of U is a subspace of X.

(Proof is a HW exercise.) □

Definition

The range of T is the image $R_T := T(X)$. The rank of T is $\dim R_T$.

The nullspace (or "kernel") of T is the preimage of 0:

$$N_T := T^{-1}(0) = \{x \in X \mid Tx = 0\}\,.$$

The nullity of T is $\dim N_T$.

Remark

A linear map $T\colon X \to U$ is 1–1 if and only if $N_T = \{0\}$.

The rank-nullity theorem

Theorem 2.2

Let $T\colon X \to U$ be a linear map. Then $\dim R_T + \dim N_T = \dim X$.

Proof

Consequences of the rank-nullity theorem

Corollary A

Suppose $\dim U < \dim X$. Then $Tx = 0$ for some $x \neq 0$.

Proof

Example A

Take $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, with $m < n$. Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be any linear map (see Example (vi)).

Since $m = \dim U < \dim X < n$, Corollary A implies that the system of m equations

$$\sum_{j=1}^{n} t_{ij} x_j = 0 \qquad i = 1, \ldots, m$$

has a non-trivial solution, i.e., not all $x_j = 0$.

Consequences of the rank-nullity theorem

Corollary B

Suppose $\dim X = \dim U < \infty$ and the only vector satisfying $Tx = 0$ is $x = 0$. Then $R_T = U$.

Proof

Example B

Take $X = U = \mathbb{R}^n$, and $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $\sum_{j=1}^{n} t_{ij}x_j = u_i$, for $i = 1, \dots, n$.

If the related homogeneous system of equations $\sum_{j=1}^{n} t_{ij}x_j = 0$, for $i = 1, \dots, n$, has only the trivial solution $x_1 = \cdots = x_n = 0$, then the inhomogeneous system T has a unique solution for any choice of $u_1 \dots, u_n$.

[*Reason*: $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism.]