Lecture 2.5: The transpose of a linear map

Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 8530, Advanced Linear Algebra

Overview

Every undergraduate linear algebra student learns about the transpose of a matrix, formed by flipping it across its main diagonal.

But what does this *really* represent?

The transpose of a matrix is what results from swapping rows with columns.

In our setting, we like to think about vectors in X as column vectors, and dual vectors in X' as row vectors.

The transpose is a more general concept than just an operation on matrices.

Given a linear map $T\colon X \to U$, its transpose is a certain induced linear map $T'\colon U' \to X'$ between the dual spaces.

In the next lecture, we will learn how to encode linear maps with matrices. When we do this, the matrix of the transpose map will simply be the tranpose of the matrix.

In this lecture, we will introduce the transpose of a linear map and study some of its basic properties.

Definition and properties

Let $T\colon X \to U$ be linear and $\ell \in U'$.

The composition $m := \ell T$ is a linear map $X \to K$.

Since T is fixed, this defines a linear map, called the transpose of T:

$$T'\colon U' \longrightarrow X', \qquad T'\colon \ell \longmapsto m,$$

$$\begin{tikzcd} X \arrow[d, "T"'] \arrow[dr, dotted, "m"] & \\ U \arrow[r, "\ell"'] & K \end{tikzcd}$$

Using scalar product notation we can rewrite $m(x) = \ell(T(x))$ as $(m, x) = (\ell, Tx)$.

Key property

The transpose of $T\colon X \to U$ is the (unique) map $T'\colon U' \to X'$ that satisfies $m = T'\ell$, i.e.,

$$(T'\ell, x) = (\ell, Tx), \qquad \text{for all } x \in X,\ \ell \in U'.$$

Caveat: We are writing ℓT for $\ell \circ T$, but $T'\ell$ for $T'(\ell)$ (much like Tx for $T(x)$).

Properties (HW exercise)

Whenever meaningful, we have

$$(ST)' = T'S', \qquad (T+R)' = T' + R', \qquad (T^{-1})' = (T')^{-1}.$$

Systems of equations

Examples (cont.)

(xi) Let $X = \mathbb{R}^N$, $U = \mathbb{R}^M$, and $Tx = u$, where $u_i = \displaystyle\sum_{j=1}^{N} t_{ij} x_j$.

$$\begin{tikzcd}
X = \mathbb{R}^N \arrow[d, "T"'] \arrow[dr, dotted, "m"] & \\
& K = \mathbb{R} \\
U = \mathbb{R}^M \arrow[ur, "\ell"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[d, mapsto, "T"'] \arrow[dr, mapsto, "m"] & \\
& (\ell, u) = (m, x) \\
u \arrow[ur, mapsto, "\ell"'] &
\end{tikzcd}$$

By definition, for some $\ell_1, \dots, \ell_m \in K$,

$$(\ell, u) = \sum_{i=1}^{M} \ell_i u_i = \sum_{i=1}^{M} \ell_i \left(\sum_{j=1}^{N} t_{ij} x_j \right) = \sum_{i=1}^{M} \sum_{j=1}^{N} \ell_i t_{ij} x_j = \sum_{i=1}^{N} \left(\ell_i \sum_{j=1}^{M} t_{ij} x_j \right) = \sum_{j=1}^{N} m_j x_j$$

This gives us a formula for $m = (m_1, \dots, m_N)$, where $(\ell, u) = (m, x)$.

We'll see later that if we express T in matrix form, then T' is formed by making the rows of T the columns of T'.

What does this really mean?

$$(\ell, u) = \sum_{i=1}^{M} \ell_i u_i = \sum_{i=1}^{M} \ell_i \left(\sum_{j=1}^{N} t_{ij} x_j \right) = \sum_{i=1}^{M} \sum_{j=1}^{N} \ell_i t_{ij} x_j = \sum_{i=1}^{N} \left(\ell_i \sum_{j=1}^{M} t_{ij} x_j \right) = \sum_{j=1}^{N} m_j x_j$$

The nullspace of the transpose

Proposition 2.8

If X'' and U'' are canonically identified with X and U, respectively, then $T'' = T$. $\square$

Proposition 2.9

The annihilator of the range of T is the nullspace of its transpose, i.e., $R_T^\perp = N_{T'}$.

Proof

Applying $\perp$ to both sides of $R_T^\perp = N_{T'}$ (Proposition 2.9) yields the following:

Corollary 2.10

The range of T is the annihilator of the nullspace of T', i.e., $R_T = N_{T'}^\perp$. $\square$

The rank of the transpose

Theorem 2.11

For any linear mapping $T\colon X \to U$, we have $\dim R_T = \dim R_{T'}$.

Proof

Corollary 2.12

Let $T\colon X \to U$ be linear with $\dim X = \dim U$. Then $\dim N_T = \dim N_{T'}$.

Proof