Lecture 3.5: The determinant and trace of a matrix

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The determinant of a $2 \times 2$ matrix

The determinant of an $n \times n$ matrix can be thought of as an alternating $n$-linear function of its column vectors.

Let’s use bilinearity to find a formula for the determinant of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. 

The determinant of a $3 \times 3$ matrix

Let’s now apply this to finding the determinant of $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. 
The determinant of an $n \times n$ matrix

**Proposition 3.8**

The determinant of an $n \times n$ matrix $A = (a_{ij})$ is

$$\det A = \sum_{\pi \in S_n} a_{1, \pi(1)} a_{2, \pi(2)} \cdots a_{n, \pi(n)},$$

and by symmetry, $\det A = \det A^T$. 

The trace of a matrix

Definition

The trace of an \( n \times n \) matrix is \( \text{tr} A = \sum_{i=1}^{n} a_{ii} \).

Proposition 3.9

(a) Trace is linear: \( \text{tr}(kA) = k(\text{tr} A) \) and \( \text{tr}(A + B) = \text{tr} A + \text{tr} B \).

(b) Trace is “commutative”: \( \text{tr}(AB) = \text{tr}(BA) \).

(c) Similar matrices have the same determinant and trace.