Lecture 4.6: Generalized eigenspaces

Matthew Macauley

School of Mathematical & Statistical Sciences
Clemson University
http://www.math.clemson.edu/~macaule/

Math 8530, Advanced Linear Algebra

Goals

Assume K is algebraically closed, and $\dim X = n$. Last time, we proved the following:

Spectral theorem

Let $A\colon X \to X$ be linear. Then

$$X = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k},$$

where $E_{\lambda_j} = \bigcup_{m=1}^{\infty} N_{(A-\lambda_j I)^m}$ is the generalized eigenspace of λ_j.

We motivated it with a running example, a map with $p_A(t) = (t-\lambda)^{11}$, and $\dim N_{A-\lambda I} = 4$:

$$\begin{array}{ccccccccccc}
v_5 & \xmapsto{A-\lambda I} & v_4 & \xmapsto{A-\lambda I} & v_3 & \xmapsto{A-\lambda I} & v_2 & \xmapsto{A-\lambda I} & v_1 & \xmapsto{A-\lambda I} & 0 \\
 & & & & w_3 & \xmapsto{A-\lambda I} & w_2 & \xmapsto{A-\lambda I} & w_1 & \xmapsto{A-\lambda I} & 0 \\
 & & & & & & x_2 & \xmapsto{A-\lambda I} & x_1 & \xmapsto{A-\lambda I} & 0 \\
 & & & & & & & & y_1 & \xmapsto{A-\lambda I} & 0
\end{array}$$

However, we haven't actually proven that the generalized eigenvectors have this structure. In this lecture, we will show how to explicitly construct such a basis.

We'll also see why the generalized eigenspace structure determines the similarity class of A.

Generalized eigenspaces characterize similarity

Let $A\colon X \to X$ have eigenvalue λ of degree d_λ. For each $m = 1, 2, \dots,$ define

$$N_m(\lambda) = N_{(A-\lambda I)^m}, \qquad \text{and note that} \quad E_\lambda = \bigcup_{m=1}^{\infty} N_m(\lambda).$$

It turns out that A (up to a choice of basis) is completely determined by the dimensions of these "eigen-subspaces" $N_1(\lambda), \dots, N_{d_\lambda}(\lambda)$, for each λ.

For another $B\colon X \to X$ with eigenvalue λ, denote its eigen-subspaces by $M_m(\lambda) = N_{(B-\lambda I)^m}$.

Theorem 4.11

The linear maps A and B are similar if and only if for each eigenvalue λ,

$$\dim N_m(\lambda) = \dim M_m(\lambda), \qquad \text{for all } m = 1, 2, \dots.$$

The "$\Rightarrow$" implication is easy. Let $A = PBP^{-1}$.

Then $(A - \lambda I)^m = P(B - \lambda I)^m P^{-1}$, and similar maps have the same nullity.

For the "$\Leftarrow$" implication, we need to construct a basis for E_λ under which $A - \lambda I$ (and hence $B - \lambda I$) admits a nice matrix form.

This is the Jordan canonical form.

Basis construction (algebraic description)

Lemma 4.7 (HW)

The map $A-\lambda I$ is a well-defined injective map on quotient spaces, i.e.,

$$A-\lambda I\colon N_{j+1}/N_j \hookrightarrow N_j/N_{j-1}, \qquad\qquad A-\lambda I\colon \bar{x} \longmapsto \overline{(A-\lambda I)x}.$$

Therefore, $\dim(N_{j+1}/N_j) \leq \dim(N_j/N_{j-1})$.

We will construct our basis in batches, from "left-to-right", starting with $N_d = E_\lambda$.

Let $\bar{x}_1, \dots, \bar{x}_{\ell_0}$ be a basis for N_d/N_{d-1}.

Apply $A-\lambda I$, to get $(A-\lambda I)\bar{x}_j \mapsto \bar{x}'_j$.

The vectors $\bar{x}'_1, \dots, \bar{x}'_{\ell_0}$ are linearly independent in N_{d-1}/N_{d-2}. Extend to a basis $\bar{x}'_1, \dots, \bar{x}'_{\ell_1}$.

Apply $A-\lambda I$, to get $(A-\lambda I)\bar{x}'_j \mapsto \bar{x}''_j$.

The vectors $\bar{x}''_1, \dots, \bar{x}''_{\ell_1}$ are linearly independent in N_{d-2}/N_{d-3}. Extend to a basis $\bar{x}''_1, \dots, \bar{x}''_{\ell_2}$.

Repeat this process, until we reach the genuine eigenvectors. The collection of representatives we've constructed is a basis for E_λ.

Basis construction (visualization)

Key points

$$A - \lambda I \colon N_{j+1}/N_j \hookrightarrow N_j/N_{j-1} \qquad \Longrightarrow \qquad \dim(N_{j+1}/N_j) \leq \dim(N_j/N_{j-1}).$$

$$\begin{array}{ccccccccccc}
x_1 & \stackrel{A-\lambda I}{\longmapsto} & x_1' & \longmapsto & x_1'' & \longmapsto & \cdots & \longmapsto & x_1^{(d)} & \longmapsto & 0 \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \\
x_{\ell_0} & \longmapsto & x_{\ell_0}' & \longmapsto & x_{\ell_0}'' & \longmapsto & \cdots & \longmapsto & x_{\ell_0}^{(d)} & \longmapsto & 0 \\
 & & x_{\ell_0+1}' & \longmapsto & x_{\ell_0+1}'' & \longmapsto & \cdots & \longmapsto & x_{\ell_0+1}^{(d)} & \longmapsto & 0 \\
 & & \vdots & & \vdots & & & & \vdots & & \\
 & & x_{\ell_1}' & \longmapsto & x_{\ell_1}'' & \longmapsto & \cdots & \longmapsto & x_{\ell_1}^{(d)} & \longmapsto & 0 \\
 & & & & x_{\ell_1+1}'' & \longmapsto & \cdots & \longmapsto & x_{\ell_1+1}^{(d)} & \longmapsto & 0 \\
 & & & & \vdots & & & & \vdots & & \\
 & & & & x_{\ell_2}'' & \longmapsto & \cdots & \longmapsto & x_{\ell_2}^{(d)} & \longmapsto & 0 \\
 & & & & & & \ddots & & \vdots & & \\
 & & & & & & & & x_{\ell_d}^{(d)} & \longmapsto & 0
\end{array}$$