Lecture 5.1: Inner products and Euclidean structure

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Overview

Up until now, much of our previous theory has been algebraic in flavor. What's been missing is a metric.

In this section, we will study vector spaces where we also have a notion of length.

As a result, this part of the class will contain more analysis, and less algebra.

In regular Euclidean space, we have standard concepts such as length and angle.

These allow us to speak of orthogonality, and to project vectors onto other vectors, or onto subspaces.

All of this is made possible by the dot product:

$$\langle x, y \rangle := x \cdot y = (x_1, \dots, x_n) \cdot (y_1, \dots, y_n) = x_1 y_1 + \cdots + x_n y_n.$$

This works because the dot product is a symmetric bilinear form with an additional property.

In this section, we will abstract this notion to the concept of an inner product.

Throughout, we will assume that X is an n-dimensional vector space over $\mathbb{R}$.

Euclidean geometry

The length or norm of $x \in X$, denoted $||x||$, is the distance from x to $0 \in X$.

By the Pythagorean theorem, $||x|| = \sqrt{x_1^2 + \cdots + x_n^2}$. Clearly, $||x||^2 = \langle x, x \rangle$.

Since the dot product is symmetric and bilinear:

$$\begin{aligned} \langle x+y, x+y \rangle &= \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \\ &= ||x||^2 + 2\langle x, y \rangle + ||y||^2 \\ &= ||x+y||^2. \end{aligned}$$

Likewise,

$$\begin{aligned} \langle x-y, x-y \rangle &= \langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle \\ &= ||x||^2 - 2\langle x, y \rangle + ||y||^2 \\ &= ||x-y||^2. \end{aligned}$$

Remarks

- This is independent of the choice of basis (coordinate system)
- Geometrically, we understand $||x||$, $||y||$, and $||x-y||$, but not $\langle x, y \rangle$... yet.

How the dot product defines angles

To understand $\langle x, y \rangle$, we'll pick a special x and y.

Given any basis ("coordinate system") $x_1, \ldots, x_n$:

1. Let x be a scalar of x_1. Then $x = (||x||, 0, \ldots, 0)$.
2. Let $y \in \text{Span}(x_1, x_2)$. Then $y = (||y|| \cos\theta,\ ||y|| \sin\theta,\ 0, \ldots, 0)$.

The dot product of x and y is thus

$$\langle x, y \rangle = (||x||, 0, \ldots, 0) \cdot (||y|| \cos\theta, ||y|| \sin\theta, 0, \ldots, 0) = ||x||\, ||y||\, \cos\theta.$$

We can characterize the angle between x and y as

$$\cos\theta = \frac{\langle x, y \rangle}{||x||\, ||y||}.$$

We can also derive the law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

Remark

One requirement for generalizing Euclidean space will be that $-1 \leq \cos\theta \leq 1$, i.e.,

$$-1 \leq \frac{\langle x, y \rangle}{||x||\, ||y||} \leq 1.$$

Fundamental properties of Euclidean space

Cauchy-Schwarz inequality

For all $x, y \in \mathbb{R}^n$,

$$|\langle x, y \rangle| \leq ||x|| \cdot ||y||,$$

and equality holds if and only if x and y are scalar multiples of each other.

Triangle inequality

For all $x, y \in \mathbb{R}^n$,

$$||x + y|| \leq ||x|| + ||y||.$$

Corollary 5.1

For any $x \in \mathbb{R}^n$,

$$||x|| = \max \big\{ \langle x, y \rangle : ||y|| = 1 \big\}.$$

Generalizing the dot product

The dot product on $\mathbb{R}^n$ gives us a notion of:

- *length*: $||x|| = \sqrt{\langle x, x \rangle}$
- *angle*: $\cos\theta = \dfrac{\langle x, y \rangle}{||x||\,||y||}$

But there's nothing special about the dot product, other than it's a symmetric bilinear form that is additionally positive-definite:

$$\langle x, x \rangle > 0, \qquad \text{for all } x \neq 0.$$

Definition

An inner product on a real vector space X is a symmetric positive-definite bilinear form

$$\langle -, - \rangle \colon X \times X \longrightarrow \mathbb{R}.$$

A vector space endowed with an inner product is an inner product space.

Key point

Everything we've done thus far (Cauchy-Schwarz, triangle inequality, etc.) works for a general inner product spaces.

Examples & non-examples

Let's explore some examples, and see what works and what doesn't.

- $X = \mathbb{R}^2$ with inner product

$$\langle a_1e_1 + a_2e_2,\ b_1e_1 + b_2e_2\rangle = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 2a_1b_1 + a_1b_2 + a_2b_1 + 2a_2b_2.$$

- $X = \mathbb{R}^2$ with inner product

$$\langle a_1e_1 + a_2e_2,\ b_1e_1 + b_2e_2\rangle = \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1b_1 + 2a_1b_2 + 2a_2b_1 + a_2b_2.$$

- $X = \mathbb{R}^2$ with inner product

$$\langle a_1e_1 + a_2e_2,\ b_1e_1 + b_2e_2\rangle = a_1b_2 + a_2b_1.$$

- $X = \mathrm{Hom}(X, Y)$ with inner product

$$\langle A, B\rangle = \mathrm{tr}(B^TA) = \sum_{i,j} a_{ij}b_{ij}.$$

- $X = \mathcal{C}[a, b]$, the space of continuous functions $f\colon [a, b] \to \mathbb{R}$ with inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx.$$