Lecture 5.4: Adjoints

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Identifying a space with its dual

Early on, we thought of scalar functions as row vectors, because intuitively:

"*Every $\ell \in X'$ can be realized by simply taking the dot product with some fixed vector.*"

In the previous lecture, we generalized this to arbitrary (n-dimensional) inner product spaces.

Key point

Every scalar function $\ell \in X'$ can be expressed as $\langle -, y\rangle$, for some $y \in X$.

This canonically identifies X with X', via $y \mapsto \langle -, y\rangle$.

Let's compare two examples of this:

- $\mathbb{R}^2$, with $(x, y) := [y_1\ y_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1y_1 + x_2y_2$,
- $\mathbb{R}^2$, with $\langle x, y\rangle := [y_1\ y_2]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$.

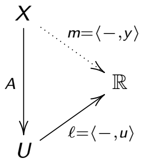
The transpose vs. the adjoint

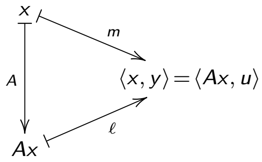
Consider a linear map $A\colon X \to U$ between real inner product spaces.

The transpose of $A\colon X \to U$ is a linear map $A'\colon U' \to X'$ satisfying

$$(A'\ell, x) = (\ell, Ax), \qquad x \in X,\ \ell \in U'.$$

In the picture below, $A'\colon \ell \mapsto m$.





If we identify X and U with their duals via $y \mapsto \langle -, y\rangle$, the transpose $\langle -, u\rangle \mapsto \langle -, y\rangle$ defines a map $u \mapsto y$ called the adjoint of A, denoted A^*.

Key idea

Given a linear map $A\colon X \to U$,

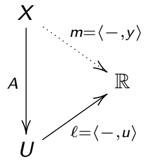
- the transpose $A'\colon U' \to X'$ maps $\ell \mapsto m$, independent of an inner product,
- the adjoint $A^*\colon U \to X$ maps $u \mapsto y$, and depends on the inner product structure.

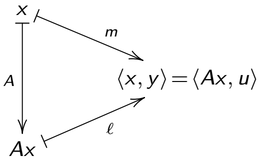
Formal definition of the adjoint

Definition

Let $A\colon X \to U$ be a linear map between real inner product spaces. The adjoint of A is the unique map $A^*\colon U \to X$ such that

$$\underbrace{\langle x, A^* u\rangle}_{\text{inner product in } X} = \underbrace{\langle Ax, u\rangle}_{\text{inner product in } U}.$$





Basic properties of adjoints

Proposition 5.7

Let $A, B\colon X \to U$ and $C\colon U \to V$ be linear maps between real inner product spaces.

(i) $(A+B)^* = A^* + B^*$

(ii) $(CA)^* = A^*C^*$

(iii) If A is bijective, then $(A^{-1})^* = (A^*)^{-1}$

(iv) $(A^*)^* = A$

(v) The matrix representations of A and A^* are transposes of each other.

Adjoints and the four subspaces

Proposition 5.8 (HW)

Let $A\colon X \to U$ be a linear maps between finite-dimensional inner product spaces. Then

(a) $N_{A^*} = R_A^\perp$

(b) $R_{A^*} = N_A^\perp$

(c) $N_A = R_{A^*}^\perp$

(d) $R_A = N_{A^*}^\perp$.

Together, this tells us that

- $X = R_{A^*} \oplus N_A$ "*the orthogonal complement of the row space is the nullspace*"
- $U = R_A \oplus N_{A^*}$ "*the orthogonal complement of the column space is the left nullspace*"