Lecture 5.6: Isometries

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Overview

Roughly speaking, an isometry is a distance-preserving map.

Definition

Let X be an inner product space. A function $A\colon X \to X$ is an isometry if

$$||Ax - Ay|| = ||x - y||, \qquad \text{for all } x, y \in X.$$

Examples

The following are all isometries of $\mathbb{R}^n$:

1. any translation
2. any rotation
3. any reflection
4. any compositions of these.

The isometries of X form a group ... but that's not a group we're all that interested in.

Orthogonal maps

Given any isometry, one can compose it with a translation to get an isometry that fixes 0.

Conversely, *any* isometry can be decomposed into one that fixes 0, followed by a translation.

Definition

An isometry $A \colon X \to X$ fixing 0 is said to be orthogonal.

The orthogonal maps on X form a group called the orthogonal group, denoted $O(X)$.

If $X = \mathbb{R}^n$, we denote this by $O(n)$ or O_n.

We will say that a matrix orthogonal if it represents an orthogonal linear map.

Remark

A matrix A is orthogonal if and only if its columns are orthonormal. That is, if $A^T A = I$.

Next, we'll show that all orthogonal maps are linear.

Properties of orthogonal maps

Theorem 5.13

Let $A \colon X \to X$ be orthogonal.

(i) A is linear

(ii) $A^*A = I$ (and conversely)

(iii) A is invertible, and A^{-1} is an isometry

(iv) $\det A = \pm 1$.

Key point

The geometric meaning of this theorem is that any map fixing 0 that preserves distances is linear, preserves angles, and preserves volume.

Definition

The subgroup of $O(X)$ of maps with determinant 1 is the special orthogonal group, denoted $SO(X)$.

Elements in $SO(X)$ describe rotations.