Lecture 7.3: Polar decomposition

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The idea of the polar decomposition

Every nonzero complex number $z \in \mathbb{C}$ has a unique polar form

$$z = re^{i\theta} = |z|e^{i\theta}, \qquad r \in \mathbb{R}^+, \quad \theta \in [0, 2\pi).$$

This can be thought of as decomposing $z \in \mathbb{C}$ into:

- a rotation by θ,
- a scaling by $|z| = r = \sqrt{\bar{z}z}$.

This is simply the polar decomposition of a 1×1 matrix.

Every linear map $A \in \mathrm{Hom}(X, X)$ can be decomposed as $A = UP$, where

- U is unitary; i.e., an isometry of X,
- $P \geq 0$; a scaling along an orthonormal axis $u_1, \ldots, u_n$.

It turns out that $P = \sqrt{A^*A} := |A|$, and so sometimes this is written $A = U|A|$.

In this lecture, we will derive the polar decomposition of a linear map

$$A \colon X \longrightarrow U, \qquad \dim X = m, \quad \dim U = n.$$

In the next lecture, we will derive the celebrated singular value decomposition (SVD).

Singular values

Key properties (Propositions 7.2, 7.6)

- $A^*A \geq 0$;
- Every $P \geq 0$ has a unique nonnegative square root $R := \sqrt{P}$, such that $R^2 = P$.

This means that for some $\lambda_1, \dots, \lambda_m \geq 0$,

$$A^*A = W \begin{bmatrix} \lambda_1^2 & & \\ & \ddots & \\ & & \lambda_m^2 \end{bmatrix} W^*, \qquad \text{and} \qquad \sqrt{A^*A} = W \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix} W^*.$$

Definition

The eigenvalues of $\lambda_1, \dots, \lambda_m$ of $\sqrt{A^*A}$ are called the singular values of A.

Facts (that we've seen)

- $||Ax|| = \left|\left|\sqrt{A^*A}x\right|\right|$ for all $x \in X$.
- A, A^*A, and $\sqrt{A^*A}$ have the same nullspace.
- A, A^*A, and $\sqrt{A^*A}$ have the same rank.

Polar decomposition of an invertible map

Theorem

Every linear map $A\colon X \to U$ can be written as $A = UP$ where $P \geq 0$ and U is unitary. This is called the (left) polar decomposition of A.

To construct the polar decomposition, suppose $A = UP$.

Since $P \geq 0$, we can write $P = QDQ^*$,and so

$$P^*P = (QDQ^*)^*(QDQ^*) = (QD^*Q^*)QDQ^* = QD^2Q^* = P^2.$$

Now, notice that

$$A^*A = (UP)^*(UP) = P^*U^*UP = P^*P = P^2.$$

Therefore, $P = \sqrt{A^*A}$.

If A is invertible, then $U = AP^{-1} = A\sqrt{A^*A}^{-1}$ is uniquely determined.

In this case,

$$A = UP = \big(A\sqrt{A^*A}^{-1}\big)\sqrt{A^*A}.$$

If A is not invertible, then U still exists, but is not unique.

Polar decomposition of an general linear map

Theorem

Every linear map $A\colon X \to U$ can be written as $A = UP$ where $P \geq 0$ and U is unitary. This is called the polar decomposition of A.

Suppose the eigenvalues of $\sqrt{A^*A}$ are

$$\lambda_1 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = \lambda_m = 0,$$

and pick a set $x_1, \ldots, x_m$ of orthonormal eigenvectors. Then

$$\frac{1}{\lambda_1}Ax_1, \ldots, \frac{1}{\lambda_r}Ax_r, x_{r+1}, \ldots, x_m$$

is orthonormal. The polar decomposition is $A = UP$ where $P = \sqrt{A^*A}$ and

$$U = \begin{bmatrix} | & & | & | & & | \\ \frac{1}{\lambda_1}Ax_1 & \cdots & \frac{1}{\lambda_r}Ax_r & x_{r+1} & \cdots & x_m \\ | & & | & | & & | \end{bmatrix} \begin{bmatrix} \text{---} & x_1^H & \text{---} \\ & \vdots & \\ \text{---} & x_m^H & \text{---} \end{bmatrix}.$$

Remark

If $A\colon X \to X$ and $r := \det P = |\det A|$, then

$$\det A = \det U \, \det P = e^{i\theta} \cdot r.$$