Lecture 7.6: Monotone matrix functions

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The square root of a positive map

Last time, we learned about the symmetrized product $AB + BA$ of self-adjoint maps, and proved the following:

Proposition 7.11

Let A, B be self-adjoint. If $A > 0$ and $AB + BA > 0$, then $B > 0$.

Corollary 7.12

If $0 < M < N$, then $0 < \sqrt{M} < \sqrt{N}$.

Examples of monotone matrix functions

Examples

Let's investigate which of the following are mmfs:

1. $f(t) = t^{-1}$
2. $f(t) = \sqrt{t}$
3. $f(t) = t^2$
4. $f(t) = t^{-2^k}$
5. $f(t) = \ln t$

Examples of monotone matrix functions

It is clear that a positive multiple, sums, or limits of mmfs is an mmf.

For $m_j, s_j > 0$, the following is an mmf:

$$f(t) = -\sum_{j=1}^{n} \frac{m_j}{t + s_j}$$

So is the "continuous version" of this:

$$f(t) = at + b - \int_0^\infty \frac{dm(s)}{t+s}, \qquad a > 0,\ b \in \mathbb{R} \tag{1}$$

where $m(t)$ is any non-negative measure for which the integral converges.

Theorem (Loewner, 1934)

Every mmf has the form of Eq. (1)

Examples of monotone matrix functions

Theorem (Loewner, 1934)

Every mmf is of the form

$$f(t) = at + b - \int_0^\infty \frac{dm(t)}{t+s}, \qquad a > 0,\ b \in \mathbb{R} \tag{1}$$

where $m(t)$ is any non-negative measure for which the integral converges.

Surprisingly, functions of this form are easy to characterize.

Theorem (Herglotz, Riesz)

Every function that is analytic on the upper half-plane with $\Im(f) > 0$ there, and $\Im(f) = 0$ on the real-axis, has the form in Eq. (1).

Conversely, *every* function in Eq. (1) can be extended to be analytic on the upper half-plane with $\Im(f) > 0$ there.