Topics: Elementary counting problems, the binomial theorem, combinatorial reciprocity.

1. Without the use of Google or AI, give a combinatorial proof of the following identity:

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Specifically, come up with a problem whose enumeration is the LHS and the RHS.

2. If $r \in \mathbb{R}$ and $k \in \mathbb{N}_0$, then the generalized binomial coefficient is defined as

$$\binom{r}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Using generating functions, this leads to the well-known combinatorial reciprocity formula

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{-n}{k}(-1)^k.$$

Now, suppose that $n \in \mathbb{N}_0$ is fixed, but k is allowed to be negative. Define

$$\binom{n}{-k} = \frac{n(n+1)\cdots(n+k-1)}{k!}.$$

Does this lead to an analogous combinatorial reciprocity formula? Explore what happens if n is allowed to be negative, and then for arbitrary $n \in \mathbb{R}$.

3. Recall *Vandermode's identity*, which says that

$$\binom{n+m}{k} = \sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-1}.$$

Explore this if one or more of the parameters are negative. Find one or more combinatorial reciprocity identities, and explain what it means. Does an analogous proof with generating functions work? Show a few small examples to illustrate this.