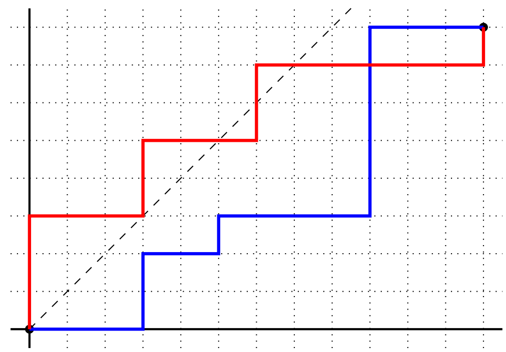
Topics: Lattice paths, generating functions, combinatorial reciprocity.

1. Consider the set of *lattice paths* in $\mathbb{Z} \times \mathbb{Z}$ that only take *positive steps*, i.e., $(1,0)$ or $(0,1)$, from the origin $(0,0)$ to (m,n), for some $m \geq n$. Two examples of these are shown below for $m = 12$ and $n = 8$.



   (a) Show that there are exactly $\binom{n+m}{n}$ such lattice paths.

   (b) Say that a lattice path is *good* if it stays below the diagonal, except possibly at its starting and ending points. All other paths are *bad*. Show that there is a bijection between the sets of
   - bad lattice paths from $(1,0)$ to (m,n),
   - all lattice paths from $(0,1)$ to (m,n).

   Then show that these sets have size $\binom{n+m-1}{m}$.

   (c) Subtract the answer from Part (b) from that of Part (a), and simplify, to get that there are
$$G(m,n) := \frac{m-n+1}{m+1}\binom{m+n}{n}$$
   good lattice paths from $(0,0)$ to (m,n).

   (d) Derive a reciprocity formula involving the number of path the touch, but don't cross, the diagonal.

2. In class, we proved the following identity:
$$\frac{1}{\sqrt{1-4x}} = \sum_{n \geq 0} \binom{2n}{n} x^n.$$

   (a) Use this to prove a related identity:
$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2(n-k)}{n-k} = 4^n.$$

   You are encouraged to use tools like ChatGPT for help; just document its usage.

(b) ChatGPT says about the above identity: "*If you prefer a combinatorial view: the left side counts ordered pairs of Dyck-path-like objects whose total size is n.*" Use tools like ChatGPT and/or Google to dive into this and explain exactly what this means. Include several examples.

3. In class, we proved that

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k,$$

where $S(n,k)$ are the Stirling numbers of the second kind, which count the number of partitions of $[n]$ into k blocks. Then, we did an explicit example to demonstrate this using $n = 3$. Carry out the details of this for $n = 4$.