Topics: Generating functions, compositions, and partitions.

1. Use Google and/or ChatGPT to find a really fun application of how generating functions can solve a particular mathematical problem. Write up a description that demonstrates that you understand all of the details.

2. In class, we derived the generating function for $p_{\leq k}(k)$, the number of partitions of n where each part is at most k. Let $c_{\leq k}(n)$ be the number of compositions where each part is at most k.

   (a) Show that $c_{\leq k}(n)$ is the coefficient of x^n in
   $$\sum_{n=0}^{\infty}(x+x^2+\cdots+x^{k-1})^n.$$

   (b) Derive the following generating function:
   $$\sum_{n=0}^{\infty}c_{\leq k}(n)x^n=\frac{1-x}{1-2x+x^k}.$$

3. In class, we constructed a bijection
   $$\big\{\text{partitions of } n \text{ into distinct parts}\big\}\longrightarrow\big\{\text{partitions of } n \text{ into odd parts}\big\}.$$
   The proof involved writing each distinct part as $2^i\cdot(2j+1)$, i.e., a power of 2 times an odd number. This map can be inverted using the binary representation of the multiplicity of each odd part. Is there a "base-3" analogue of this result? If so, what would it look like? If not, explain what fails and why.

4. In class, we used probability generating functions to prove that it is impossible to assign weights to two six-sided dice so that the probability distribution of their sum is uniform. The argument was that the product
   $$P(x)Q(x)=\frac{1}{11}\big(1+x+\cdots+x^{10}\big)=\frac{1}{11}\cdot\frac{x^{11}-1}{x-1}$$
   cannot have all ten non-real 11^{th} roots of unit as zeros.

   (a) Show how this argument carries over for n-sided dice, for all even $n\geq 4$.

   (b) Try to adapt this argument for n-sided dice, where n is odd. Does it work? The main difference is that -1 is also a $(2n-1)^{\text{th}}$ root of unity in this case.