Topics: Permutation statistics

1. Recall that $\begin{bmatrix} n \\ k \end{bmatrix} = c(n,k)$ is the number of permutations in $\mathfrak{S}_n$ with k cycles. In class, we proved that

$$\sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{cyc}(\pi)} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k = x(x+1)(x+2)\cdots(x+n-1) = x^{\overline{n}}.$$

In this problem, we will explore this for $n = 5$.

(a) Determine how many cycle types there are for in S_5, and the size of each one using elementary counting arguments.

(b) Carry out the details to verify the identity above for $n = 5$.

2. Recall the *fundamental bijection*

$$\theta \colon \mathfrak{S}_n \longrightarrow \mathfrak{S}_n, \qquad \theta \colon \pi \longmapsto \widehat{\pi},$$

that maps a permutation in its standard cycle notation to the permutation whose 1-line notation is the result of removing the parantheses.

(a) Prove that the following are equivalent:

(i) π is a product of cycles of length at most 2,

(ii) $\pi^{-1} = \pi$.

(b) Prove that if $\widehat{\pi} = \pi$, then $\pi^{-1} = \pi$.

(c) Prove that the number of such permutations is a Fibonacci number. [*Hint*: show that they satisfy the Fibonacci recurrence.]

3. In class, we proved the identity

$$\sum_{\pi \in S_n} x^{\mathrm{inv}(\pi)} = (1+x)(1+x+x^2)\cdots(1+x+x^2+\cdots+x^{n-1})$$

Carry out the details of this for $n = 4$.

4. Compute the inversion table of the permutation $w = 854192637$ in $\mathfrak{S}_9$. Then, by inserting the numbers $9, 8, \ldots, 1$ sequentially, construct the permutation $w \in \mathfrak{S}_9$ that has inversion table $I(w) = (6,3,0,2,4,1,1,1,0)$.

5. In class, we proved that the *Eulerian polynomials* satisfy

$$A_n(x) = \sum_{k=0}^{n} A(n,k)x^k = \sum_{\pi \in \mathfrak{S}_n} x^{\mathrm{des}(\pi)}.$$

Carry out the details of this for $n = 4$. That is, for each of the $4! = 24$ permutations $\pi \in \mathfrak{S}_4$, write $\mathrm{des}(\pi)$, and use this to find the polynomial $A_4(x)$.