Topics: Eulerian and Mahonian statistics

1. In class, we computed the descent, excedance, weak excedance, fixed points, inversion, and major index, for the 6 permutations in $\mathfrak{S}_3$. They are summarized by the following table and polynnomials..

w	$\widehat{w}$	$\mathrm{exc}(\pi)$	$\mathrm{wexc}(\pi)$	$\mathrm{fix}(\pi)$	$\mathrm{des}(\pi)$	$\mathrm{inv}(\pi)$	$\mathrm{maj}(\pi)$
$(1)(2)(3)$	123	0	3	3	0	0	0
$(1)(3\underline{2})$	$1\underline{3}2$	1	2	1	1	1	2
$(2\underline{1})(3)$	$\underline{2}13$	1	2	1	1	1	1
$(3\underline{12})$	$2\underline{3}1$	2	2	0	1	2	2
$(32\underline{1})$	$\underline{3}12$	1	1	0	1	2	1
$(3\underline{1})(2)$	$\underline{32}1$	1	2	1	2	3	3

$$\sum_{w\in\mathfrak{S}_3} x^{\mathrm{exc}(w)} = \sum_{w\in\mathfrak{S}_3} x^{\mathrm{des}(w)} = 1+4x+4x^2, \qquad \sum_{w\in\mathfrak{S}_3} x^{\mathrm{inv}(w)} = \sum_{w\in\mathfrak{S}_3} x^{\mathrm{maj}(w)} = 1+2x+2x^2+x^3.$$

Create a similar table for the 24 permuations in $\mathfrak{S}_4$, and write the corresponding polynomials. You may use ChatGPT to help as long as you summarize your steps and usage.

2. In class, we saw the 1978 bijection $\varphi\colon \mathfrak{S}_n \to \mathfrak{S}_n$ of Foata and Schützenberger for which $\mathrm{maj}(\pi) = \varphi(\mathrm{inv}(\pi))$. This is defined by starting with a word $w = w_1 \cdots w_n$, and recursively defining the words $\gamma_1, \dots, \gamma_n$, where each γ_i has length i, according to the following rules:

   (a) Set $\gamma_1 = w_1$.

   (b) Given γ_i:

      i. Split it into subwords:

         A. If $w_i > w_{i+1}$ split after each number $> w_{i+1}$

         B. If $w_i > w_{i+1}$ split after each number $< w_i$.

      ii. Cyclically shift each resulting compartment 1 to the right

      iii. Add w_{i+1} at the end to get γ_{i+1}

   (c) The image of w is γ_n.

   In class, we did an example to demonstrate that $\varphi(683941725) = 364891725$, and we arranged each step in a row in a table, like the following:

unsplit	$\mathrm{inv}(\gamma_i)$	w_i vs. w_{i+1}	split	cyclically shift
$\gamma_1 = 6$	0	$6 < 8$	6	6
$\gamma_2 = 68$	0	$8 > 3$	6 | 8	6 | 8
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\gamma_8 = 63894712$	18	$2 < 5$	63 | 894 | 71 | 2	36 | 489 | 17 | 2
$\gamma_9 = 364891725$	18	–	–	–

   Carry out a similar exercise for the permutation $w = 469528137$.

3. Recall the *fundamental bijection* $w \mapsto \widehat{w}$ of Foata that removes the parantheses from a permutation in standard cycle notation. In class, we considered the following permutation under the inverse map:

$$\widehat{w} = \begin{bmatrix} \color{red}{1} & 2 & \color{red}{3} & 4 & \color{red}{5} & \color{red}{6} & \color{red}{7} & 8 \\ 2 & 1 & 6 & 3 & 5 & 7 & 8 & 4 \end{bmatrix} \longmapsto (2\underline{1})(6\underline{35})(7)(8\underline{4}) = w,$$

where the (strict) excedences in $\widehat{w}$ are underlined, and the weak excedences in w are in red. Recall that these are defined as

$$\mathrm{Wexc}(w) = \{i \in [n] : w(i) \geq i\}, \qquad \mathrm{Exc}(w) = \{i \in [n] : w(i) > i\},$$

and note that 1 will *always* be a weak excedence. Show how to construct a bijection

$$\mathrm{Wexc}(\widehat{w}) \setminus \{1\} \longmapsto \mathrm{Exc}(w).$$

You are encouraged to use ChatGPT, but you must document your usage, and note that it seems to be *very* prone to making errors on this particular problem!