Fri. 8/22: Four ways to count

Mon. 8/25: The pigeonhole principle

1. Prove that there exists a positive integer n so that 44^n is divisible by 7.
2. Suppose that $A \subseteq \{1, 2, \ldots, 2n\}$, where $n \in \mathbb{N}$ and $n \geq 2$. Show that if $|A| = n+1$, then there are at least two relatively prime numbers in A.
3. Suppose that $A \subseteq \{1, 2, \ldots, 2n\}$, where $n \in \mathbb{N}$ and $n \geq 2$. Show that if $|A| = n+1$, then there must be some number in A that divides another number of A.

Wed. 8/27: Induction

4. Prove that
$$1 \cdot 2 + 2 \cdot 3 + \cdots + (n-1) \cdot n = \frac{(n-1)n(n+1)}{3}.$$

Fri. 8/29: Strong induction, basic counting

5. Use strong induction to prove that a positive integer is divisible by 3 iff the sum of its digits is divisible by 3.

Wed. 9/3: Acyclic orientations, permutations, and subsets

6. A host invites n couples to a party. She wants to ask a subset of the $2n$ guests to give a speech, but she does not want to ask both members of any couple to speak. In how may ways can she proceed?

Fri. 9/5: Generating functions for subsets. Compositions

7. How many compositions of n are there in which the first part is not 2?
8. How many compositions of n are there into four parts, each of which is at least 2?
9. How many weak compositions of 10 are there, in to five parts so that exactly two parts are zero?

Mon. 9/8: Multisets

Wed. 9/10: Binomial theorem and combinatorial reciprocity

10. Show that
$$\binom{\frac{1}{2}}{k} = \frac{(-1)^{k-1}}{k \cdot 2^{2k-1}} \binom{2k-2}{k-1}.$$
11. Use the identity in the previous problem and the generalized binomial theorem to write $\sqrt{1+x}$ as an infinite series.

Fri. 9/12: Ehrhart theory

Mon. 9/15: Multinomial coefficients

12. A sports team has to visit four cities to play a league opponent, each of them five times. In how many different ways can they do this if they are not allowed to start and finish in the same city?

Wed. 9/17: The twelvefold way

13. How many 4-digit positive integers are there in which all digits are different?

14. In how many ways can the elements of the set $[n]$ be permuted so that the sum of every two consecutive elements in the permutation is odd?

15. In how many different ways can we place eight identitcal rooks on a chess board so that no two of them attack each other?

16. In how many ways are there to select an 11-member soccer team and a 5-member basketball teacm from a class of 30 students if:

    (a) nobody can be on two teams

    (b) any number of students can be on both teams

    (c) at most one student can be on both teams?

17. How many different ways can a race with 8 runners be completed, assuming no ties?

18. A woman is about to build her own home computer. She decides she has three different possible brands of chip, four possible hard drives, three choices of keyboard, and five brands of monitor. How many different ways are there for the woman to build her computer?

19. A president, a treasurer, and a secretary are to be chosen from a committee with forty members. In how many ways could the three officers be chosen?

20. A fair 6-sided die is rolled 8 times and the resulting sequence of 8 numbers is recorded.

    (a) How many different sequences are possible?

    (b) How many different sequences consist entirely of even numbers?

    (c) How many different sequences are possible if the first, third, and fourth numbers must be the same?

21. A school dance committee is to consist of 2 freshmen, 3 sophomores, 4 juniors, and 5 seniors. If 6 freshmen, 8 sophomores, 9 juniors, and 7 seniors are eligible to be on the committee, in how many ways can the committee be chosen?

22. A boy has 3 red , 5 yellow and 2 green marbles. In how many ways can the boy arrange the marbles in a line if:

    (a) Marbles of the same color are indistinguishable?

    (b) All marbles have different sizes?

Fri. 9/19: Set partitions

23. Find a closed formula for $S(n,2)$ if $n \geq 2$.

24. Find a formula for $S(n,3)$, for $n \geq 3$. (Start with finding the number of surjections $f\colon [n] \to [3]$.)

Mon. 9/22: Integer partitions

25. Suppose that the number of self-conjugate partitions of n is even. Is $p(n)$ even or odd, and why?

26. Let $n \geq 7$. Prove that the number of partitions of n into four distinct parts is at least as large as the number of parttions of $n-6$ into four parts. Are these numbers equal?

Wed. 9/24: Generating functions for partitions

Fri. 9/26: Cycles in permutations