Appendix C: Writing Advice

Ever since you were young, you have been taught how to write. You have told stories, summarized readings and composed speeches, and your success at each of these was in-part dependent on your ability to communicate well. Some day you may write essays to get a scholarship, go to graduate school, or earn a job. It can be challenging to summarize a book you read or tell a story from your life. It can be even more challenging to communicate a complicated mathematical argument in a way that is clear and mistake-free. In this appendix I have collected some advice to improve your proof writing.

C.1 Writing Proofs

For now, be as rigorous as possible

As someone taking their first proof-based math course, you should strive to be as rigorous and thorough as possible. You should begin every proof by stating your assumptions. You should end every proof with a summarizing sentence noting that you reached the conclusion. Clearly cite every used definition and work out all of your algebraic steps. If your proof uses the fact that "an even plus an even is an even," and that has not been proven yet in this course, then you should prove it. In future courses, it will be ok to say "an even plus an even is an even" with no further justification. In this course, though, while you are learning proof methods, facts like that have to be proven or cited. This is good — when learning a new proof method, it can be helpful to prove things which are mathematically simple, to allow you to put all your focus on the method and on your proof writing.

As I said in the Chapter 3 Pro-Tips, this is how I think about it: When my dad taught me how to drive, he insisted that I do everything *perfectly.* Hand placement, mirrors, speed limit, spacing, signs, blinkers, lights, focus, radio, ... every last thing should be done perfectly. It's not that being soooo meticulous is crucial; less so would still be plenty safe. It's because everyone relaxes this alertness eventually — and if you start by driving perfectly, then once you relax you will still end up in a great place. This was my dad's reasoning.

The same reasoning holds with proof writing. If your proofs begin with surgical precision, then once you inevitably relax a bit, you won't do so to a point that mistakes are introduced or your readers are confused.[1] Moreover, even if certain aspects of your proof-writing relax, it is important that your thinking stays just as sharp. This ultra-rigorous approach will not only improve your proof-writing, but will help train your brain to be logically meticulous.

While some of the advice below is also tailored to where you are now, most of it is solid advice for your entire math career. That certainly includes this piece of timeless advice: know your audience.

Know your audience

As with all communication, what counts as "effective" proof-writing depends on who will likely be reading your work. What terminology is common knowledge, and what should be redefined for the reader? What level of rigor is expected or required?

When you are writing up a homework assignment, you can generally use all of the definitions and theorems that have been covered up to that point in the course (although you should cite them when you use them). If you are writing a research article to submit to the *AKCE International Journal of Graphs and Combinatorics*, then your audience certainly knows what a graph's vertices and edges are, and so you may omit that detail. When writing for a class, you are expected to prove more of the details than you would for a research article. Conversely, in a research article it would be more common to accompany your solutions with your own examples, which illustrate some of your proof's abstract ideas in a concrete setting.[2]

For your homework, a good rule-of-thumb is to write as if you were trying to convince a classmate.

When taking a writing class, you may have learned about writing with "voice." What tone do you want to set? If you are writing a text message, your tone is probably much more informal than if you are applying for a job. You should feel welcome to develop your own style and voice to your proof-writing, but make sure not to lose the rigor when doing so.[3]

Grammatical rules

It is common to think that rules of English (or whatever language you are writing in) are not as important when you are writing about math. This is false. You should

[1]When my grandma taught my dad to drive, her advice was simpler: "Assume every other driver is an idiot." While I did consider making this the lesson for your proof writing... I ultimately chose to go with my dad's more wholesome take.

[2]Feel free to do this in your homework, too! Just make sure it doesn't take away from the proof itself. It should be thorough without the example.

[3]If you read some other math textbooks and then read one of mine, you'll probably discover that in my view math should be much more conversational than it is in most published textbooks — especially when your audience is college students. So I hesitate to say too much right now, since the main point of this appendix is to convey the broadly-held views of mathematicians on writing, not push my own outlier views, and of course you should check with your professor when it comes to your coursework. That said, in a footnote I am comfortable saying that I think almost all math is too formal. And I hope the next generation of mathematicians moves the norm towards a lighter tone.

write in complete sentences. You should have natural paragraph breaks. You should capitalize and punctuate just as you would in a writing class.[4] If your sentence ends with a mathematical expression, it still needs a period. Example: If $|x| = 2$, then $x = 2$ or $x = -2$.

There are also some math-specific rules which should be observed. Most importantly, you should not start a sentence with a variable or symbols; having it appear after a period can be confusing to the reader. So instead of starting a sentence saying "f satisfies the intermediate value property" you could say write something like "The function f satisfies the intermediate value property." Usually the way around it is to add a couple of words stating what the variable/symbol means. Two more examples: instead of starting a sentence with "$\mathbb{Q}$ is a subset of $\mathbb{R}$" or "$x^2 - 2x + 1 = 0$ has one solution," you could write "The set $\mathbb{Q}$ is a subset of $\mathbb{R}$" or "The equation $x^2 - 2x + 1 = 0$ has one solution."

When writing a long equation with many parts or indices, it often wise to place it in its own line. So instead of saying "notice that $\sum_{n=1}^{\infty} \int_0^n \cos^n(x) \frac{x^n}{n}\, dx = f_2(x, y)$," you would write it like this: "notice that

$$\sum_{n=1}^{\infty} \int_0^n \cos^n(x) \frac{x^n}{n}\, dx = f_2(x, y)."$$

Giving it its own line allows the details to be larger and situated better. (Also, notice that the expression ended a sentence and hence ended with a period.)

Finally, when you define a term, you should *italicize* it, underline it, or make it **bold**. In this book, I italicized my terms when I defined them, but bold is also common. In handwritten work, underlining is common.

Using symbols

On a chalkboard or in your scratch work, it is common to use symbols like $\forall$ for "for all," $\exists$ for "there exists," $\Rightarrow$ for "implies," and $\Leftrightarrow$ for "if and only if." You may use abbreviations like iff for "if and only if," wlog for "without loss of generality," and s.t. for "such that." These allow you to write things quickly, keeping your attention on the math.[5] But in formal writing, they should not be used.

Incidentally, mathematicians have agreed that *some* symbols are always ok. For example, you may use numbers (-2, 3, 43.4534, $\frac{2}{3}$, etc.), you may use numbers which have been given their own symbol (π, e, φ, etc.), you may use the standard arithmetic symbols ($+$, $-$, $\cdot$) and variable names (a, n, x, etc.), you may use symbols for named sets ($\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$) and the standard set symbols ($\cup$, $\cap$, $\setminus$, $\in$). Therefore, you may write "there exists $n \in \mathbb{N}$," but you should not write "$\exists n$ in the set $\mathbb{N}$."

[4] One exception that I have adopted: If I want to refer to many versions of, say, the variable k, it can confuse the reader to write "the ks all have the property...," since this looks like k is multiplied by s. Thus, at times I break the apostrophe/pluralization rule and write "the k's all have the property...." This is not done widely, but if you must decide between clarity and grammar, and there is no good way to satisfy both, then at times it may be reasonable to side with clarity. (As you'll see, the main times I break from convention is when I am concerned about clarity.)

[5] And writing things like "$\forall x \in \mathbb{N}, \exists y \in \mathbb{R}$, s.t. $x^2 = y$" lets you feel like you know how to write in a second language. And that second language is hieroglyphics.

We also allow (and embrace) the use of two particular abbreviations: i.e. (which means "that is") and e.g. (which means "for example").[6]

Understandably, these rules may seem arbitrary and hard to remember. But with a little experience reading books/articles/websites on proof-based math, you will quickly pick up these customs.[7] In closing this section, here are a few more small tips when working with symbols:

- Make sure you do not introduce a new symbol without defining it in some way. In some cases, a symbol x could be defined as simply as this: "Note that there is some $x \in \mathbb{N}$ such that $x > 100$." This has given x a meaning. In other cases, you may formally define what a set S is. Every symbol without a known definition (like $\mathbb{N}$) must be introduced in some way.

- While using a symbol like $\subseteq$ is acceptable when writing, say, "Note that $\{1, 2\} \subseteq \{1, 2, 3\}$, which implies...," it would be improper to write "Since our first set is $\subseteq$ our second set...." You should also not write things like "Note that the two sets are =." Those symbols can be use as parts of a mathematical expression, but not as simple word-replacements.

- This is a perfectly clear English sentence: "Because it snowed last night, I have to shovel the driveway." Here is a very similar math sentence which is less clear: "Because $x^2 = 4$, $x = 2$ or $x = -2$." The comma makes it look like it is part of a list. Plus, we use commas for other purposes in math,[8] so scrunching one between two math symbols could confuse your reader on their first read — is that a grammatical comma or a mathematical comma? Thus, instead of "Because $x^2 = 4$, $x = 2$ or $x = -2$," you could write "Because $x^2 = 4$, we see that $x = 2$ or $x = -2$." Or, "Because $x^2 = 4$, we have $x = 2$ or $x = -2$." Or, "Because $x^2 = 4$, it follows that $x = 2$ or $x = -2$." There are many options, but it is advised to insert a couple words to avoid confusion.

Don't say too little

A common difficulty is deciding how much you should include in your proof. This is especially difficult in an Intro to Proofs class — if you have to say why the sum of two evens is even, do you also have to say why the sum of two integers is an integer? The sum of two real numbers is a real number? Is there anything you *don't* have to justify? In later classes, when you are allowed to assume the basic properties of arithmetic, things are a little easier, but still it can be a challenge. The best general advice that I can give is to check with your professor when you are unsure. Instead, let's talk about general math writing tips.

[6] Grammar Pro-Tip: Always add a comma after you use "e.g." or "i.e." in a sentence. E.g., the sentence "An even times an even is an even; e.g., $3 \cdot 4 = 12$."

[7] Learning customs is important, too. This book is teaching you what it is like to be a mathematician, and our customs are part of it.

[8] Main examples: in ordered pairs, like $(2, 3)$; in a function which has multiple arguments, like $f(x, y)$; in sets appearing as a list, like $\{1, 2, 3, \dots\}$; or intervals, like the closed interval $[a, b]$.

In most writing classes, your teachers tell you that your writing should be short, punchy, to-the-point. If you can use one word instead of two, do it. If you can use one sentence instead of two, *definitely* do it. In math, the advice is not so straightforward. To begin, you should make sure to say enough that your solution is complete. You should cite every definition and theorem you use — and you should say why they apply. You should work out any algebra that is needed, and you should ensure that it is clear how the logic takes the reader from one step to the next. On this point, students usually err on the side of justifying too little than justifying too much.

That said, it is certainly possible for a proof to be too wordy.[9] If you can directly say what you mean in a clear and concise way, then do it. But oftentimes in math, clarity is the cost of concision. Every mathematician is familiar with the experience of having to read a single sentence three or four[10] times to understand it. Instead of one confusing sentence, if the author could have explained the same idea in two clear sentences which the reader would have understood without having to reread them, then using one sentence is *less* efficient for your readers than using two. You should optimize your writing for your readers, not for the word-counter.

In general, my advice is to first explain your reasoning so that it is clear and complete, regardless of how much writing it takes. Then, look back over your proof and see if you can find ways to make it concise without sacrificing its clarity or correctness. Usually you can! Often, you can reduce its length significantly. Having several drafts of your work before your final write-up is quite common; try to embrace the fact that writing is a sequence of small steps to the final product, and it never comes out perfectly at the start.[11]

This is an art, not a science, and practice will go a long way. Try to put yourself in your readers' shoes, and try to spot the moments where confusion could arise. Or, better yet, let your peers read over your work to see if they understand it. Thoughtful feedback is invaluable. Also, read over your peers' work to see how they proved the same result. You can pick up tips from each other, and then everyone improves.

Avoid extraneous statements

Related to being too wordy, you should avoid saying things that have no bearing on your solution. Suppose you are trying to prove that if c is a rational number, then c^2 is also rational. Suppose you start your proof like this:

> Assume that c is rational; then, by the definition of a rational number, $c = \frac{m}{n}$ for some integers m and n where $n \neq 0$. Furthermore, we may assume that m and n are chosen so that $\frac{m}{n}$ is written in lowest terms.

Everything you said is correct, but if you worked out the rest of the proof (basically, $c = \frac{m}{n}$ implies $c^2 = \frac{m^2}{n^2}$ which is again a ratio of integers and $n^2 \neq 0$), at no point did you use the fact that $\frac{m}{n}$ was written in lowest terms. Since that was never used, you should remove it. It was true, but not needed.

[9] And that's coming from a guy who is vigorously promoting "long-form" math.

[10] or five or six or ...

[11] I wish you could see how many small edits it took to arrive at the final version of this paragraph!

When one reads a confusing proof with extraneous statements, it can feel like trying to solve a messy puzzle. You want your proofs to be clear, not a jumbled mess to decipher. Can you imagine trying to solve a 500-piece jigsaw puzzle which includes 700 pieces in it, and you have to identify and remove the 200 extraneous pieces in order to complete the puzzle? You don't want your proofs to feel like that. Include no red herrings in your story.

Break up complicated proofs into parts

If a proof is longer than a couple paragraphs, it is easy for a reader to get lost in your argument. Thus, it often helps to highlight the big structure of the problem.

For example, in our induction proofs we indicated clearly when we were doing our base case, inductive hypothesis, induction step and conclusion (e.g., the proof of Proposition 4.4); in our proofs by cases, we indicated clearly when we were doing Case 1, Case 2, etc. (e.g., the proof of Proposition 2.7); in some of our proofs of if-and-only-if statements we indicated clearly when we were doing the forward direction and the backward direction (e.g., the proofs of Proposition 6.3 and Theorem 9.5); and in some of our more complicated proofs, we broke up the argument into parts, and explained how those parts fit together (e.g., the proof of Theorem 2.13).

There were also many times where the first or second paragraph of the proof was simply an explanation of what was to come, containing the big picture without any of the details (e.g., the proof of Lemma 9.10). This foreshadowing can really help the reader understand what you are doing and why.

Don't stop with your first draft

I've said it before, but it is worth repeating: Writing is a process, and your first draft should not be your last draft. Look back over your work and search for ways to improve it. Was your notation optimal? Is each part clear enough? Is there a sentence that was never used and can be removed? Could it be reorganized or restructured in a way to make it clearer?

Miscellaneous Advice

- When using a word which has a mathematical definition, take extra care that you are using it correctly. You must say what you mean, and mean what you say.[12]

- In math, it is common to use "we" when writing proofs, rather than "I." You will see passages like, "by plugging in $x = 2$, we see that f is discontinuous." Research articles with a single author will even use "we," even though it is clear they did the work by themselves. This can help create an invitational tone, where you are bringing the reader along with you.[13] Even if you solved a homework problem by yourself, you may use "we."

[12] Say it if and only if you mean it.
[13] Note: Most STEM fields do not do this.

- When possible, avoid using "it" in your writing. When you have used several nouns up to that point, it can be hard to know what "it" is referring to. For example, suppose I wrote "Since $f'(x) = g(x)$, it must be continuous." What does "it" refer to here? The function f or the function g? There are times when the reader could work harder to deduce what "it" must be referring to, but don't make them do so. Make it easy on your readers. In our example, it would be better to write "Since $f'(x) = g(x)$, we see that f is continuous."

- Mathematicians love short words to begin a new sentences. Examples: Therefore, thus, hence, consequently, however. It is good to mix up which of these you use. Doing so doesn't change the math, but it makes for a more engaging read for your readers. We even use "and" and "but" to start a sentence more than in most writing, because ending a sentence can help the reader segment off an idea in their head, separating it more clearly from the idea that you are sticking in the next sentence.

- When you took geometry in high school, you may have written proofs as two columns, where on the left you wrote a single sentence or statement, and on the right you justified it. You should not do this. All proof-based math is written in paragraph form.

- Did I mention that clarity is the most important thing? These rules are here to help you write clearly. If they get in the way of clarity, you may break them.

- Communication of all kinds is cultural, and this includes math communication. I have shared some generally-adopted practices of math writing, but these practices may depend on where you work or study, and who is reading your proofs. Be aware and responsive to the culture in which your work lives.

- It is good to thank those who contributed to your work. Books have editorial boards, and research articles have acknowledgments sections.[14] Indeed, the idea for this very bullet point came from the article *Some Guidelines for Good Mathematical Writing*, authored by Francis Su and published in the newsmagazine *MAA Focus*.

C.2 Writing in LaTeX

All formal math writing is done in a program[15] called LaTeX. This book was written in LaTeX, every other math book you have read was likely written in LaTeX, and the exams your math professors have given you were very likely written in LaTeX. It

[14]Deciding who to acknowledge can be difficult. In an age where there are so many resources that you can pull ideas from, it is impossible to thank every article, YouTube video, talk, tweet or personal conversation which may have inspired a sentence here or a paragraph there, but you should at least cite those whose work you directly use or those who reviewed your work. If you are wondering what level of assistance on your homework deserves an acknowledgment... ask your professor.

[15]To be pedantic, it is technically a "language" or a "software system."

is used for every math journal in the world, both for the submissions and the final journal publication. I once read a blog post from the editor of a math research journal, who wrote about how many crank math papers they receive each year. He said that if they read every paper from someone claiming to have solved some centuries-old conjecture, they would spend all their time on such nonsense. How do they quickly spot the submissions that can be immediately discarded? Their number 1 rule: If it is not written in LaTeX, discard it.[16]

In fact, LaTeX is nearly universal throughout the STEM fields. I once even received an email from a professor of deaf studies at my university who was writing a paper for a deaf studies journal which only accepted papers written in LaTeX. She was pretty knowledgeable already, but she had some technical questions for me about the software.

It is well worth your time to learn LaTeX. I first learned it as an undergrad. One semester I chose one of my math classes and decided to type up all of my homework for that class in LaTeX. This was a lot of extra work, as it takes time to learn. Yet with just one semester of this, I was already typing up most of my homework about as fast as it would have taken to handwrite it. From that point on, I typed up all my homework.[17] This time investment paid off big in graduate school, when LaTeX proficiency is necessary. I encourage you to try it out.

For instructions on how to write in LaTeX, and templates that you can use for your homework, check out this book's page on the `LongFormMath.com` website.

[16]No matter how good Microsoft Equation Editor gets, it will forever send shudders down the backs of mathematicians of a certain age.

[17]Which is still stored on my computer to this day! Mistakes and all!