

Math 4120

Study guide: Midterm 1.

Note: This is just a guide, not an all-inclusive list.

Definitions to know.

- (1) A *group* G . (The “official” definition.)
- (2) The *order* of an element $g \in G$.
- (3) A *left coset* xH of a subgroup $H \leq G$.
- (4) A *normal subgroup* $H \trianglelefteq G$.
- (5) The *index* $[G : H]$ of a subgroup $H \leq G$.
- (6) The *direct product* $A \times B$ of two groups A and B .
- (7) The *quotient* G/H of a group G by a normal subgroup $H \trianglelefteq G$.
- (8) The *normalizer* $N_G(H)$ of a subgroup $H \leq G$.
- (9) The *center* $Z(G)$ of a group.
- (10) What it means for multiplication $aH \cdot bH := abH$ in the quotient group G/H to be *well-defined*.
- (11) The *conjugacy class* $\text{cl}_G(x)$ of an element $x \in G$, and the conjugacy class $\text{cl}_G(H)$ of a subgroup.
- (12) The *centralizer* $C_G(x)$ of an element $x \in G$.

Cayley diagrams and presentations.

- (1) Be able to use a Cayley diagram as a “group calculator”, e.g., multiply elements and find their inverses.
- (2) Be able to construct Cayley diagrams of V_4 , C_n , D_n , Q_8 , Dic_n , SD_8 , SA_8 , and write a group presentation for these groups.
- (3) Given an unknown Cayley diagrams, write a group presentation that describes it.
- (4) Be able to identify left and right cosets from a Cayley diagram.
- (5) Be able to find the normalizer of a subgroup from a Cayley diagram.

Cycle diagrams.

- (1) Be able to construct a cycle diagram of groups like C_n , D_n , and Q_8 .
- (2) Use a cycle diagram to identify all cyclic subgroups, the first step in constructing a subgroup lattice.

Subgroup lattices.

- (1) Be able to construct the subgroup lattices of $\mathbb{Z}_n$, V_4 , $D_3 \cong S_3$, D_4 , D_5 , Q_8 , $\mathbb{Z}_4 \times \mathbb{Z}_2$, and A_4 .
- (2) Be able to label the edges of a subgroup lattice with the index, $[H : K]$.
- (3) Know how to be “fluent” reading subgroup lattices. For example, given H and K , where to find $H \cap K$ and $\langle H \cup K \rangle$, and how to identify when a subgroup is normal (e.g., G , $\{e\}$, index-2 subgroups, and unicorns).
- (4) Be able to determine the normalizer of H on a Cayley diagram, given knowledge of its conjugacy class, or vice-versa.

Helpful misc. facts about familiar groups.

- (1) The cyclic group C_n is generated by r^k , iff $\gcd(n, k) = 1$.
- (2) $C_n \times C_m \cong C_{nm}$ iff $\gcd(n, m) = 1$.
- (3) Every subgroup of Q_8 is normal.
- (4) The dihedral group D_n has n or $n + 1$ elements of order 2, depending on the parity of n . It can be generated by a rotation and reflection, or two adjacent reflections.
- (5) The dihedral group D_n is a semidirect product $C_n \rtimes_{\theta} C_2$.

- (6) There is one frieze group that needs three symmetries to generate it. It contains three non-abelian frieze groups (the “infinite dihedral group”) as subgroups: (i) removing all horizontal reflections, (ii) remove all 180° -rotations, or (iii) remove half of each of these.
- (7) Know how to represent the groups V_4 , C_n , D_n , Q_n , and Dic_n with 2×2 matrices.
- (8) Two canonical generating sets for the symmetric group: $S_n = \langle (12), (123 \cdots n) \rangle = \langle (12), (23), \dots, (n-1 \ n) \rangle$.
- (9) Know the difference between *minimal* and *minimum* generating sets.
- (10) The automorphism group $\text{Aut}(C_n)$ (of “rewirings”) is isomorphic to the group

$$U_n = \{k \mid 1 \leq k < n, \gcd(n, k) = 1\}.$$
- (11) Know how to construct the Cayley diagram of $\text{Aut}(C_n)$, and a semidirect product, given a “labeling map” $\theta: H \rightarrow \text{Aut}(C_n)$.

Useful facts and techniques.

- (1) Two different ways to show that a subset $H \subseteq G$ is a subgroup.
- (2) Three different ways to show that a subgroup $H \leq G$ is normal.
- (3) Know how to compose permutations in cycle notation, and find inverses, e.g., $(123 \cdots n)^{-1} = (1n \cdots 32)$.
- (4) Know which permutations are even vs. odd.
- (5) Learn to classify all finite abelian groups of a fixed order.
- (6) Two elements in S_n are conjugate iff they have the same cycle type.
- (7) If n is odd, then all reflections in D_n are conjugate. If n is even, then there are two conjugacy classes of reflections.
- (8) $\text{cl}_G(x) = \{x\}$ if and only if $x \in Z(G)$.
- (9) $\text{cl}_G(H) = \{H\}$ if and only if $H \trianglelefteq G$.
- (10) Use the fact that $|\text{cl}_G(x)| = [G : C_G(x)]$ to help partition G by conjugacy classes, and/or find the centralizer.
- (11) Use the fact that $|\text{cl}_G(H)| = [G : N_G(H)]$ to help partition G ’s subgroups by conjugacy classes, and/or find the normalizer.

Proofs to learn.

- (1) Show that the identity element of a group is unique.
- (2) Show that every element in a group has a unique inverse.
- (3) Show that if $\{H_\alpha \mid \alpha \in A\}$ is a collection of subgroups, then $\bigcap_{\alpha \in A} H_\alpha$ is a subgroup.
- (4) Show that $xH = H$ if and only if $x \in H$.
- (5) Show that if $[G : H] = 2$, then $H \trianglelefteq G$.
- (6) Prove that if $K \leq H \leq G$ and $K \trianglelefteq G$, then $K \trianglelefteq H$.
- (7) Show that the center $Z(G) = \{z \in G \mid gz = zg, \forall g \in G\}$ is a subgroup of G and that it is normal.
- (8) Let $H \trianglelefteq G$. Prove that multiplication of cosets is well-defined: if $a_1H = a_2H$ and $b_1H = b_2H$, then $a_1H \cdot b_1H = a_2H \cdot b_2H$. Additionally, show that G/H is a group under this binary operation.
- (9) The tower law: $[G : H][H : K] = [G : K]$.
- (10) Prove that if G is abelian and $H \leq G$, then G/H is abelian.
- (11) Show that the normalizer $N_G(H) = \{g \in G \mid gHg^{-1} = H\}$ is a subgroup of G .
- (12) Show that if $A, B \leq G$, and A normalizes B , then AB is a subgroup of G .

Study guide: Midterm 2.

Definitions to know.

- (1) A *homomorphism* ϕ from a group G to a group H .
- (2) What it means for a homomorphism to be an *embedding* and a *quotient*.
- (3) An *isomorphism* $\phi: G \rightarrow H$.
- (4) An *automorphism* $\phi: G \rightarrow H$.
- (5) The *kernel* of a homomorphism $\phi: G \rightarrow H$.
- (6) What it means for a map $f: G/N \rightarrow H$ to be *well-defined*.
- (7) The *commutator subgroup* G' of a group G , and the *abelianization* G/G' .
- (8) An *inner automorphism* and *outer automorphism* of G .
- (9) A *group action* of G on a set S .
- (10) Local features of an action: the *orbit* $\text{orb}(s)$ and *stabilizer* $\text{stab}(s)$ of $s \in S$, and the *fixator* $\text{fix}(g)$ of $g \in G$.
- (11) Global features of an action: the set $\text{Fix}(\phi)$ of *fixed points*, and the *kernel* $\text{Ker}(\phi)$.
- (12) A *p-group*, and a *Sylow p-subgroup* of a group G .

Useful facts and techniques.

- (1) Be able to show that a certain map is a homomorphism, using the definition.
- (2) A homomorphism is 1-to-1 iff $\text{Ker}(\phi) = \{1\}$.
- (3) There are two ways to prove that $G/N \cong H$: Either construct a map $G/N \rightarrow H$ and prove it is a well-defined bijective homomorphism, or construct a map $\phi: G \rightarrow H$ and prove it is an onto homomorphism with $\text{Ker}(\phi) = N$.
- (4) Learn the statement of the correspondence theorem: there is a 1–1 correspondence between subgroup of G/N and subgroups of G that contain N . Moreover, every subgroup of G/N is of the form H/N for some $N \leq H \leq G$. Be able to interpret this visually in terms of subgroup lattices.
- (5) Be able to recognize subgroups and quotients of a group simply from the subgroup lattice: subgroups appears as “stammites”, and quotients as “stalactites.”
- (6) Learn how to identify the commutator subgroup of G and abelinization G/G' just from the subgroup lattice.
- (7) The automorphism group of a cyclic group is $\text{Aut}(\mathbb{Z}_n) \cong U_n$, the multiplitive group of integers modulo n .
- (8) Inner automorphism have the form $\varphi_g: x \mapsto gxg^{-1}$. The inner automorphism group of G is $\text{Inn}(G) \cong G/Z(G)$. That is, $\varphi_g = \varphi_h$ iff g and h are in the same cosets of $Z(G)$.
- (9) Given only a subgroup lattice of G , be able to determine whether G is isomorphic to the semidirect product, or direct product, of two of its subgroups.
- (10) The orbit-stabilizer theorem: If G acts on S , then $|G| = |\text{orb}(s)| \cdot |\text{stab}(s)|$ for any $s \in S$.
- (11) The orbit counting theorem: the average size of $\text{fix}(g)$ is the number of orbits.
- (12) Learn the local featuers (orbits, stabilizers, fixed point sets), and global features (kernel, set of fixed points) for each of the following actions: following actions:
 - (i) G acting on itself by right multiplication.
 - (ii) G acting on itself by conjugation.
 - (iii) G acting on its subgroups by conjugation.
 - (iv) G acting on its right cosets by right multiplication.
- (13) Constructing the “fixed point table” of an action, and identifying the features of an action from it.
- (14) Learn how to use the 3rd Sylow theorem to show that a group of a certain order is simple. (Usually, by showing that $n_p = 1$ for some prime p .)

Proofs to learn.

- (1) If $\phi: G \rightarrow H$ is a homomorphism, then $\phi(1_G) = 1_H$.
- (2) If $\phi: G \rightarrow H$ is a homomorphism, then $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in G$.

- (3) If G is abelian, then so is G/H .
- (4) If $G/Z(G)$ is cyclic, then G is abelian (and hence $G/Z(G)$ is the trivial group).
- (5) The kernel of any homomorphism is a subgroup, and is normal.
- (6) Given a homomorphism $\phi: G \rightarrow H$, each preimage $\phi^{-1}(h)$ is a coset of $\text{Ker}(\phi)$.
- (7) $A \times B \cong B \times A$.
- (8) If $H \leq G$, then $xHx^{-1} \cong H$ for any $x \in G$.
- (9) There is no embedding $\varphi: \mathbb{Z}_n \rightarrow \mathbb{Z}$.
- (10) If $\varphi: G \rightarrow H$ is a homomorphism and $N \trianglelefteq H$, then $\varphi^{-1}(N)$ is a normal subgroup of G .
- (11) If $H \leq G$ is the only subgroup of G of order $|H|$, then it must be normal.
- (12) The FHT: if $\phi: G \rightarrow H$ is a homomorphism, then $G/\text{Ker}(\phi) \cong \text{Im}(\phi)$.
- (13) The correspondence theorem: every subgroup of G/N has the form H/N , for some $H \leq G$ that contains N .
- (14) The freshman theorem: given a chain $N \leq H \leq G$ of normal subgroups of G , $(G/N)/(H/N) \cong G/H$.
- (15) The diamond isomorphism theorem: if A normalizes G , then $AB \leq G$, $B \trianglelefteq AB$, $(A \cap B) \trianglelefteq A$, and $AB/B \cong A/(A \cap B)$.
- (16) Use the FHT to show that $|NH| = |N| \cdot |H|/|N \cap H|$.
- (17) Show that $\mathbb{Q}^* \cong \mathbb{Q}^+ \times C_2$ and $\mathbb{Q}^*/\langle -1 \rangle \cong \mathbb{Q}^+$, where $\mathbb{Q}^*$ is the nonzero rationals under multiplication, and $\mathbb{Q}^+ \leq \mathbb{Q}^*$ is the subgroup of positive rationals.
- (18) Show that G is abelian iff its commutator subgroup $G' = \{e\}$.
- (19) Show that G/G' is abelian.
- (20) Show that $\text{Inn}(G)$ is a normal subgroup of $\text{Aut}(G)$.
- (21) Use the FHT to show that $G/Z(G) \cong \text{Inn}(G)$.
- (22) Show that if G acts on S , then $\text{stab}(s)$ is a subgroup of G , for any $s \in S$.
- (23) Show that if G is a p -group, then $|Z(G)| > 1$.
- (24) Show how Cayley's theorem follows from the orbit-stabilizer theorem, and a group acting on itself by multiplication.
- (25) Show that if G has no subgroup of index 2, then any subgroup of index 3 is normal.
- (26) Show that if $[G : H] = p$ for the smallest prime dividing $|G|$, then $H \trianglelefteq G$.

Study guide: Final exam.

Note: This is *in addition*, not instead, of the Midterm 1 and 2 material.

Definitions to memorize.

- (1) A *ring* R .
- (2) A *unit*, and a *zero divisor* of a ring.
- (3) An *ideal* of a ring R (left, right, and two-sided).
- (4) Types of rings: integral domain, division ring, field.
- (5) The *quotient ring* R/I for some two-sided ideal I , and how to add and multiply elements.
- (6) A *homomorphism* ϕ from a ring R to a ring S .
- (7) A *maximal ideal* and a *prime ideal* of a ring R .

Useful facts and techniques.

- (1) Construct the subring lattice of a small finite ring, and be able to determine the ideals, subrings that aren't ideals, and subgroups that aren't subrings.
- (2) Know examples of ideals, subrings that aren't ideals, and subgroups that aren't subrings, in various rings.
- (3) Know examples of both maximal ideals and prime ideals, prime ideals that aren't maximal.
- (4) Learn how to construct a finite field $\mathbb{F}_q$ of order $q = p^k$.
- (5) Know the statements of the fundamental homomorphism theorem and the correspondence theorem for rings and how to apply them.

Proofs to learn.

- (1) If an ideal I of R contains a unit, then $I = R$.
- (2) The FHT for rings: if $\phi: R \rightarrow S$ is a ring homomorphism, then $\text{Ker}(\phi)$ is an ideal of R and $R/\text{Ker}(\phi) \cong \text{Im}(\phi)$.
- (3) Prove the isomorphism theorems for rings, assuming the results for groups.
- (4) The following are equivalent for commutative rings: (i) I is a maximal ideal, (ii) R/I is simple, (iii) R/I is a field.
- (5) An ideal P is prime iff R/P is an integral domain.
- (6) A ring R is an integral domain iff 0 is a prime ideal.
- (7) Every maximal ideal is prime.
- (8) Use Zorn's lemma to show that every ideal is contained in a maximal ideal.