1. A tank initially contains 100 gal of pure water. Water begins entering a tank via two pipes: through pipe A at 6 gal per minute, and pipe B at 4 gal per minute. Simultaneously, a drain is opened at the bottom of the tank through which solution leaves the tank at a rate of 8 gal per minute.

   (a) To their dismay, supervisors discover that the water coming into the tank through pipe A is contaminated, containing 0.5 lb of pollutant per gallon of water. If the process had been running undected for 10 minutes, how much pollutant is in the tank at the end of this 10-minute period?

   (b) The supervisors correct their error and shut down pipe A, allowing pipe B and the drain to function in precisely the same matter as they did before the contaminant was discovered in pipe A. How long will it take the pollutant in the tank to reach one half of the level achieved in Part (a)?

2. The population of a certain planet is believed to be growing according to the logistic equation. The maximum population the planet can hold is 10^{10}. In year zero the population is 50% of this maximum, and the rate of increase of the population is 10^9 per year.

   (a) What is the logistic equation satisfied by the population, $y(t)$?

   (b) How many years until the population reaches 90% of the maximum?

   (c) Sketch this solution curve in the ty-plane, as well as the steady-state solutions $y(t) = 0$ and $y(t) = 10^{10}$.

3. A lake, with volume $V = 100$ km^3, is fed by a river at a rate of r km^3/yr. In addition, there is a factory on the lake that introduces a pollutant into the lake at the rate of p km^3/yr. There is another river that is fed by the lake at a rate that keeps the volume of the lake constant. This means that the rate of flow from the lake into the outlet river is $(p+r)$ km^3/yr. Let $x(t)$ denote the volume of the pollutant in the lake at time t. Then $c(t) = x(t)/V$ is the concentration of the pollutant.

   (a) Show that, under the assumption of immediate and perfect mixing of the pollutant into the lake water, the concenetration satisfies the differential equation

   $$c' + \frac{p+r}{V}\, c = \frac{p}{V}\,.$$

   (b) It has been determined that a concentration of over 2% is hazardous for the fish in the lake. Suppose that $r = 50$ km^3/yr, $p = 2$ km^3/yr, and the initial concentration of pollutant in the lake is zero. How long will it take the lake to become hazardous to the health of the fish?

   (c) Suppose that the factory from parts (a) and (b) stops operating at time $t = 0$, and that the concentration of pollutant in the lake was 3.5% at that time. Approximately how long will it take before the concentration falls below 2% and the lake is no longer hazardous for the fish? Assume that river still flows out at a rate the leaves the lake volume constant.

4. Consider two tanks, labeled tank A and tank B for reference. Tank A contains 100 gal of solution in which is dissolved 20 lb of salt. Tank B contains 200 gal of solution in which is dissolved 40 lb of salt. Pure water flows into tank A at a rate of 5 gal/sec. There is a drain at the bottom of tank A, and the solution leaves tank A via this drain at a rate of 5 gal/sec and flows immediately into tank B at the same rate. A drain at the bottom of tank B allows the solution to leave tank B at a rate of 2.5 gal/sec. What is the salt content in tank B at the precise moment that tank B contains 250 gal of solution.

5. For each of the second-order differential equations below, decide whether the equation is linear or nonlinear. If the equation is linear, state whether the equation is homogeneous or inhomogeneous.

   (a) $y'' + 3y' + 5y = 3\cos 2t$
   (b) $t^2y'' = 4y' - \sin t$
   (c) $t^2y'' + (1-y)y' = \cos t$
   (d) $ty'' + (\sin t)y' = 4y - \cos 5t$
   (e) $t^2y'' + 4yy' = 0$
   (f) $y'' + 4y' + 7y = 3e^{-t}\sin t$
   (g) $y'' + 3y' + 4\sin y = 0$
   (h) $(1-t^2)y'' = 3y$

6. Find the general solution to the following 2^{nd} order linear homogeneous ODEs.

   (a) $y'' + 5y' + 6y = 0$
   (b) $y'' + y' - 12y = 0$
   (c) $y'' + 4y' + 5y = 0$
   (d) $y'' + 2y = 0$
   (e) $y'' - 4y' + 4y = 0$
   (f) $4y'' + 12y' + 9y = 0$

7. In this problem, we will find all solutions to the *boundary value problem* (BVP) $y'' = \lambda y$, $y(0) = y(\pi) = 0$, where λ is a constant. This equation will turn up later when we study PDEs.

   (a) First, suppose that $\lambda = 0$. That is, solve $y'' = 0$, $y(0) = y(\pi) = 0$.
   (b) Next, suppose $\lambda = \omega^2 \geq 0$.
      (i) Solve the boundary value problem $y'' = \omega^2 y$, $y(0) = y(\pi) = 0$.
      (ii) Let $u_1(t) = \cosh \omega t = \frac{e^{\omega t}+e^{-\omega t}}{2}$ and $u_2(t) = \sinh \omega t = \frac{e^{\omega t}-e^{-\omega t}}{2}$. Show that $u_1(t)$ and $u_2(t)$ both solve $y'' = \omega^2 y$, and use this to write the general solution of this differential equation.
      (iii) Solve the boundary value problem from part (i) again, but this time, start by using the general solution you found in Part (ii) (instead of exponentials).

(c) Finally, suppose $\lambda = -\omega^2 < 0$. That is, solve $y'' = -\omega^2 y$, $y(0) = y(\pi) = 0$.

(d) Using your results from parts (a)–(c), describe all solutions to the boundary value problem $y'' = \lambda y$, $y(0) = y(\pi) = 0$. What are the possibile values for λ?

8. Sove the following initial value problems.

(a) $y'' - y' - 2y = 0, \quad y(0) = -1, \quad y'(0) = 2$

(b) $y'' - 4y' - 5y = 0, \quad y(1) = -1, \quad y'(1) = -1$

(c) $y'' + 25y = 3, \quad y(0) = 1, \quad y'(0) = -1$

(d) $y'' - 2y' + 17y = 0, \quad y(0) = -2, \quad y'(0) = 3$

9. Find the general solution to the following 2^{nd} order linear inhomogeneous ODEs, by solving the associated homogeneous equation, and then finding a particular (constant) solution.

(a) $y'' + y' - 12y = 24$

(b) $y'' = -4y + 3$

10. As we've seen, to solve ODE of the form

$$y'' + py' + qy = 0\,, \qquad p \text{ and } q \text{ constants}$$

we assume that the solution has the form e^{rt}, and then we plug this back into the ODE to get the *characteristic equation*: $r^2 + pr + q = 0$. Given that this equation has a double root $r = r_1$ (i.e., the roots are $r_1 = r_2$), show by direct substitution that $y = te^{rt}$ is a solution of the ODE, and then write down the general solution.

11. Suppose that $z(t) = x(t) + iy(t)$ is a solution of

$$z'' + pz' + qz = Ae^{i\omega t}\,.$$

Substitute $z(t)$ into the above equation. Then compare (equate) the real and imaginary parts of each side to prove two facts:

$$x'' + px' + qx = A\cos\omega t$$
$$y'' + py' + qy = A\sin\omega t\,.$$

Write a sentence or two summarizing the significance of this result.

12. Solve the following initial value problems using the method of undetermined coefficients.

(a) $y'' + 3y' + 2y = -3e^{-4t}, \quad y(0) = 1, \quad y'(0) = 0$

(b) $y'' + 2y' + 2y = 2\cos 2t, \quad y(0) = -2, \quad y'(0) = 0$

(c) $y'' + 4y' + 4y = 4 - t, \quad y(0) = -1, \quad y'(0) = 0$

(d) $y'' - 2y' + y = t^3, \quad y(0) = 1, \quad y'(0) = 0$

13. Solve the following first order differential equations using the method of undetermined coefficients.

(a) $y' - 3y = 5e^{2t}$

(b) $y' + 2y = 3t$

(c) $T' = k(-t - T)$

(d) $T' = k(A \sin \omega t - T)$