1. If $y_f(t)$ is a solution of
$$y'' + py' + qy = f(t)$$
and $y_g(t)$ is a solution of
$$y'' + py' + qy = g(t)\,,$$
show that $z(t) = \alpha y_f(t) + \beta y_g(t)$ is a solution of
$$y'' + py' + qy = \alpha f(t) + \beta g(t)\,,$$
where α and β are any real numbers, by plugging it into the ODE.

2. Find the general solution to the following 2^{nd} order linear inhomogeneous ODEs.
   (a) $y'' + 2y' + 2y = 2 + \cos 2t$
   (b) $y'' + 25y = 2 + 3t + 4\cos 2t$
   (c) $y'' - y = t - e^{-t}$.

3. (a) Find the general solution of $y'' + 3y' + 2y = te^{-4t}$. (Look for a particular solution of the form $y_p = (at + b)e^{-4t}$.)
   (b) Use a similar approach as above to find a solution to the differential equation $y'' + 2y' + y = t^2 e^{-2t}$.

4. Find the general solution of $y'' + 2y' + 2y = e^{-2t}\sin t$. (Look for a particular solution of the form $y_p = e^{-2t}(a\cos t + b\sin t)$.)

5. For the following exercises, rewrite the given function in the form
$$y = A\cos(\omega t - \phi) = A\cos\left(\omega\left(t - \frac{\phi}{\omega}\right)\right)\,,$$
and then plot the graph of this function.
   (a) $y = \cos 2t + \sin 2t$
   (b) $y = \cos t - \sin t$
   (c) $y = \cos 4t + \sqrt{3}\sin 4t$
   (d) $y = -\sqrt{3}\cos 2t + \sin 2t$.

6. Consider the undamped oscillator
$$mx'' + kx = 0, \qquad x(0) = x_0, \qquad x'(0) = v_0.$$
   (a) Write the general solution of this initial value problem in the form $x(t) = a\cos\omega t + b\sin\omega t$ (i.e., determine a, b, and ω.).
   (b) Write your solution in the form $x(t) = A\cos(\omega t - \phi)$ (i.e., determine A).

7. A 0.1-kg mass is attached to a spring having a spring constant 3.6 kg/s^2. The system is allowed to come to rest. Then the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4 m/s. If there is no damping present, find the amplitude A and frequency ω of the resulting motion.

   (a) Let $x = 0$ be the position of the spring *before* the mass was hung from it. Find $x(0)$.

   (b) Solve this initial value problem and plot the solution.

8. A spring-mass system is modeled by the equation

$$x'' + \mu x' + 4x = 0\,.$$

   (a) Show that the system is critically damped when $\mu = 4$ kg/s.

   (b) Suppose that the mass is displaced upward 2 m and given an initial velocity of 1 m/s. Use a computer (i.e., WolframAlpha) to comute the solution for $\mu = 4$, 4.2, 4.4, 4.6, 4.8, 5. Plot all of the solution curves on one figure. What is special about the critically damped solution in comparison to the other solutions?

   (c) On a new set of axes, repeat part (b) using $\mu = 4$, 3.9, and 3.

   (d) Explain why would you want to adjust the spring on a screen door so that it was critically damped.

9. The function $x(t) = \cos 6t - \cos 7t$ has mean frequency $\bar{\omega} = 13/2$ and half difference $\delta = 1/2$. Thus,

$$\cos 6t - \cos 7t = \cos\left(\frac{13}{2} - \frac{1}{2}\right)t - \cos\left(\frac{13}{2} + \frac{1}{2}\right)t = 2\sin\frac{1}{2}t\ \sin\frac{13}{2}t\,.$$

   Plot the graph of $x(t)$, and superimpose the "envelope" of the beats, which is the slow frequency oscillation $y(t) = \pm 2\sin(1/2)t$. Use different line styles or colors to differentiate the curves.

10. Plot the given function on an appropriate time interval. Use the technique of the previous exercise to superimpose the plot of the envelope of the beats in a different line style and/or color.

    (a) $\cos 9t - \cos 10t$

    (b) $\sin 11t - \sin 10t$

11. Let $\omega_0 = 11$. Use a computer to plot the graph of the solution

$$x(t) = \frac{\cos\omega t - \cos\omega_0 t}{\omega_0^2 - \omega^2}$$

    for $\omega = 9$, 10, 10.5, 10.9, and 10.99 on the time interval $[0, 24]$. (Okay to just print this out and attach it). Explain how these solutions approach the resonance solution as $\omega \to \omega_0$. [*Hint*: Put the equation above in the form $x(t) = A\sin\delta t\ \sin\bar{\omega}t$, and use this result to justify your conclusion.]

12. For each system below, write it as $\mathbf{A}\mathbf{x} = \mathbf{b}$. Find all solutions, and sketch the graph of the lines in each system on the same axis. Are the resulting lines intersecting, parallel, or coincident?

    (a) $x_1 + 3x_2 = 0, \quad 2x_1 - x_2 = 0$

    (b) $-x_1 + 2x_2 = 4, \quad 2x_1 - 4x_2 = -6$

    (c) $2x_1 - 3x_2 = 4, \quad x_1 + 2x_2 = -5$

    (d) $3x_1 - 2x_2 = 0, \quad -6x_1 + 4x_2 = 0$

    (e) $2x_1 - 3x_2 = 6, \quad -4x_1 + 6x_2 = -12$

13. For each part, find the determinant, eigenvalues and eigenvectors of the given matrix. If the matrix is invertible, find its inverse.

    (a) $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$ (c) $\mathbf{A} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

    (d) $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}$ (e) $\mathbf{A} = \begin{pmatrix} -1 & -4 \\ 1 & -1 \end{pmatrix}$ (f) $\mathbf{A} = \begin{pmatrix} 5/4 & 3/4 \\ -3/4 & -1/4 \end{pmatrix}$

14. For each problem below, find the eigenvalues of the given matrix, and then describe how the nature of the eigenvalue (e.g., positive/negative, complex, repeated, etc.) depends on the parameter α.

    (a) $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & \alpha \end{pmatrix}$ (b) $\mathbf{A} = \begin{pmatrix} 1 & -\alpha \\ 2\alpha & 3 \end{pmatrix}$

15. Let A be a 2×2 matrix. In this problem we will show that $\lambda = 0$ is an eigenvalue of a matrix $\mathbf{A}$ if and only if $\det(\mathbf{A}) = 0$.

    (a) Write the characteristic polynomial (i.e., the polynomial $\det(\mathbf{A} - \lambda I) = 0$ in terms of the determinant and trace of A.

    (b) Show that if $\lambda = 0$ is an eigenvalue of $\mathbf{A}$, then $\det(\mathbf{A}) = 0$.

    (c) Show that if $\det(\mathbf{A}) = 0$, then $\lambda = 0$ is an eigenvalue of $\mathbf{A}$.

    (d) Now, make the same argument – that $\lambda = 0$ is an eigenvalue if and only if $\det(\mathbf{A}) = 0$, without reference to the characteristic polynomial. (*Hint*: If $\lambda = 0$ is an eigenvalue, then $\mathbf{A}\mathbf{v} = 0\mathbf{v} = \mathbf{0}$ for some $\mathbf{v} \neq \mathbf{0}$. When does such a homogeneous system have a non-zero solution?)