1. Express the y-intercept of $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$ in terms of the a_n's and b_n's. (*Hint*: It's *not* a_0 or $a_0/2$!)

2. Consider the 2π-periodic function $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$. Write the Fourier series for the following functions:

   (a) The reflection of $f(x)$ across the y-axis;

   (b) The reflection of $f(x)$ across the x-axis;

   (c) The reflection of $f(x)$ across the origin.

3. (a) The Fourier series of an odd function consists only of sine-terms. What additional symmetry conditions on f will imply that the sine coefficients with even indices will be zero (i.e., each $b_{2n}=0$)? Give an example of a non-zero function satisfying this additional condition.

   (b) What symmetry conditions on f will imply that the sine coefficients with odd indices will be zero (i.e., each $b_{2n+1}=0$)? Give an example of a non-zero function satisfying this additional condition.

   (c) Sketch the graph of a non-zero even function, such that $a_{2n}=0$ for all n.

   (d) Sketch the graph of a non-zero even function, such that $a_{2n+1}=0$ for all n.

4. Consider the function defined on the interval $[0,\pi]$:

$$f(x)=\begin{cases} x & \text{for } 0\le x<\pi/2, \\ \pi - x, & \text{for } \pi/2 \le x \le \pi. \end{cases}$$

   (a) Sketch the even extension of this function and find its Fourier cosine series.

   (b) Sketch the odd extension of this function and find its Fourier sine series.

5. Consider the function defined on the interval $[0,\pi]$:

$$f(x)=x(\pi-x).$$

   (a) Sketch the even extension of this function and find its Fourier cosine series.

   (b) Sketch the odd extension of this function and find its Fourier sine series.

6. (a) Find the complex Fourier coefficients of the function

$$f(x)=x^2 \qquad \text{for } -\pi<x\le\pi,$$

   extended to be periodic of period 2π.

   (b) Find the real form of the Fourier series. *Hint: Use* $a_n=c_n+c_{-n}$, *and* $b_n=i(c_n-c_{-n})$.

7. Compute the complex Fourier series for the function defined on the interval $[-\pi, \pi]$:

$$f(x) = \begin{cases} -1, & -\pi \le x < 0, \\ 4, & 0 \le x \le \pi. \end{cases}$$

Use the c_n's to find the coefficients of the real Fourier series (the a_n's and b_n's).

8. Find the real and complex Fourier series for the function defined on the interval $[-\pi, \pi]$:

$$f(x) = \begin{cases} 0, & -\pi \le x < 0, \\ 1, & 0 \le x \le \pi. \end{cases}$$

Only compute one of these directly (your choice), and then use the formulas relating the real and complex coefficients to compute the other.

9. Compute the complex Fourier series for the function $f(x) = \pi - x$ defined on the interval $[-\pi, \pi]$. Use the c_n's to to find the coefficients of the real version of the Fourier series.

10. Prove Parseval's identity:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\, dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \,.$$

11. Use Parseval's identity, and the Fourier series of the function $f(x) = x^2$ on $[-\pi, \pi]$, to compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

12. Compute $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$. *Hint*: Compute the Fourier series for $f(x) = |x|$, and then look at $f(\pi)$. (Parseval's identity not needed!)