1. Consider the ODE $y'' = 4y$. We know that the general solution is $y(t) = C_1e^{2t} + C_2e^{-2t}$, i.e., $\{e^{2t}, e^{-2t}\}$ is a *basis* for the solution space. Use the fact that $e^{2t} = \cosh 2t + \sinh 2t$ and $e^{-2t} = \cosh 2t - \sinh 2t$, and that any linear combination of solutions is a solution, to find two distinct solutions involving hyperbolic sines and cosines. Write the general solution using these functions.

2. We will solve for the function $u(x,t)$, defined for $0 \leq x \leq \pi$ and $t \geq 0$, which satisfies the following conditions:

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0, \qquad u(x,0) = 5\sin x + 3\sin 2x.$$

   (a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of both boundary conditions (called *Dirichlet* boundary conditions) and the initial condition.

   (b) Assume that $u(x,t) = f(x)g(t)$. Find u_t and u_{xx}. Also, determine the boundary conditions for $f(x)$ (at $x = 0$ and $x = \pi$) from the boundary conditions for $u(x,t)$.

   (c) Plug $u = fg$ back into the PDE and divide both sides by c^2fg (i.e., "separate variables") to get the *eigenvalue problem*. Briefly justify why this quantity must be a constant. Call this constant λ. Write down two ODEs: one for $g(t)$ and one for $f(x)$.

   (d) Solve for $g(t)$, $f(x)$, and λ.

   (e) Using your solution to Part (d) and the principle of superposition, find the general solution to the boundary value problem.

   (f) Solve the *initial value problem*, i.e., find the particular solution $u(x,t)$ that additionally satisfies $u(x,0) = 5\sin x + 3\sin 2x$.

   (g) What is the steady-state solution, i.e., $\lim_{t\to\infty} u(x,t)$?

3. Consider a similar situation as the previous problem, but with slightly different boundary and initial conditions.

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = 30, \quad u(\pi,t) = 100$$
$$u(x,0) = 30 + \tfrac{70}{\pi}x + 5\sin x + 3\sin 2x.$$

   (a) Describe (and sketch) a physical situation that this models. Be sure to describe the impact of *both* boundary conditions and the initial condition.

   (b) Use your physical intuition to determine what the steady-state solution $u_{ss}(x)$ is.

   (c) Write down the solution to this initial/boundary value problem by adding the steady-state solution to the solution of the related homogeneous problem (see Part (f) of the previous problem).

(d) How does this compare to the structure of the solution to the ODE for Newton's law of heating / cooling? [*Hint*: Consider an example, e.g., $T(t) = 72 + T_h(t) = 72 + Ce^{-kt}$. Note that the heat equation is the 1-dimensional analog of Newtons law of heating / cooling (which is typically applied to a point-mass, or a "0-dimensional" object).]

4. Consider the following initial/boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = 0, \quad u_x(\pi, t) = 0, \qquad u(x,0) = 3\sin\frac{5x}{2}.$$

(a) Describe (and sketch) a physical situation that this models. Be sure to describe the impact of *both* boundary conditions and the initial condition.

(b) Assume that there is a solution of the form $u(x,t) = f(x)g(t)$. Find u_t, u_x, and u_{xx}. Also, determine the boundary conditions for $f(x)$ (at $x = 0$ and $x = \pi$) from the *mixed boundary conditions* for $u(x,t)$.

(c) Plug $u = fg$ back into the PDE and divide both sides by $c^2 fg$ (i.e., "separate variables") to get the eigenvalue problem. Write down two ODEs: one for $g(t)$ and one for $f(x)$.

(d) Solve the ODEs from the previous part for f and g. You may assume that $\lambda = -\omega^2$, (i.e., that $\lambda < 0$). Determine ω (be sure to show your work for this part, the answer may surprise you!).

(e) Write down the general solution $u(x,t)$ for the boundary value problem.

(f) Find the particular solution for $u(x,t)$ that additionally satisfies the initial condition $u(x,0) = 3\sin(5x/2)$.

(g) What is the steady-state solution?

5. Let $u(x,t)$ be the temperature of a bar of length 10, at position x and time t (in hours). Suppose that the left endpoint of the bar is not insulated, but the right endpoint is fully insulated, and the bar is sitting in a 70° room. Moreover, suppose that initially, the temperature increases linearly from 70° at the left endpoint to 80° at the other end. Finally, suppose the interior of the bar is poorly insulated, so heat can escape.

(a) Suppose that heat escapes at a constant rate of 1° per hour. Write an initial/boundary value problem for $u(x,t)$ that could model this situation.

(b) A more realistic situation would be for heat to escape not at a constant rate, but at a rate proportional to the *difference* between the temperature of the bar and the ambient temperature of the room. Write an initial/boundary value problem for $u(x,t)$ that could model this situation. What is the steady-state solution and why?

6. Let $u(x,t)$ be the temperature of a bar of length 10, that is insulated so that no heat can enter or leave. Suppose that initially, the temperature increases linearly from 70° at one endpoint, to 80° at the other endpoint.

(a) Sketch the initial heat distribution on the bar, and express it as a function of x.

(b) Write down an initial/boundary value problem to which $u(x,t)$ is a solution (Let the constant from the heat equation be c^2).

(c) What will the steady-state solution be?

7. Consider the following PDE:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t)=0, \quad u_x(\pi,t)+\gamma\, u(\pi,t)=0, \qquad u(x,0)=h(x)\ ,$$

where γ is a constant, and $h(x)$ and arbitrary function on $[0,\pi]$.

(a) Describe a physical situation that this models. Be sure to describe the impact of the initial condition, *both* boundary conditions and the constant γ.

(b) What is the steady-state solution, and why? (Use your physical intuition).

8. We will solve the heat equation with *periodic boundary conditions*:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial \theta^2}, \qquad u(\theta+2\pi,t)=u(\theta,t), \qquad u(\theta,0)=2+4\sin 3\theta-\cos 5\theta.$$

(a) Describe and sketch a situation that this models.

(b) Assume that there is a solution of the form $u(\theta,t)=f(\theta)g(t)$. Find u_t and $u_{\theta\theta}$. Use the periodic boundary conditions for $u(\theta,t)$ to derive similar periodic boundary conditions for $f(\theta)$.

(c) Plug $u=fg$ back into the PDE and divide both sides by c^2fg (i.e., "separate variables") to get the eigenvalue problem. Write down two ODEs: one for $g(t)$ and one for $f(\theta)$.

(d) Solve for $g(t)$, $f(\theta)$, and λ. *Note: You won't be able to conclude that* $a=0$ *or* $b=0$ *– so unlike before, they'll both stick around.*

(e) Find the general solution of the boundary value problem. As before, it will be a superposition (infinite sum) of solutions $u_n(\theta,t)=f_n(\theta)g_n(t)$.

(f) Find the particular solution to the initial value problem that satisfies the initial condition $u(\theta,0)=2+4\sin 3\theta-\cos 5\theta$.

(g) What is the steady-state solution? Give a mathematical *and* intuitive (physical) justification for this.