1. Find the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies the following initial value problem of the *wave equation*:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = 8\sin x + 11\sin 2x + 15\sin 4x, \qquad u_t(x,0) = 0.$$

   (a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of both boundary conditions and both initial conditions.

   (b) Assume that $u(x,t) = f(x)g(t)$. Find u_t, u_{tt}, and u_{xx}. Also, determine two boundary conditions for $f(x)$ (at $x = 0$ and $x = \pi$) from the boundary conditions for $u(x,t)$, and one boundary condition for $g(t)$.

   (c) Plug $u = fg$ back into the PDE and divide both sides by $c^2 fg$ (i.e., "separate variables") to get the eigenvalue problem. Write down two ODEs: one for $g(t)$ and one for $f(x)$.

   (d) Write down the solution of the ODE for $f(x)$, and λ (this is the same as in the heat equation; there is no need to re-derive it). Solve the ODE for $g(t)$.

   (e) Using your solution to (d) and the principle of superposition, find the general solution to the initial/boundary value problem. As before, it will be a superposition (infinite sum) of solutions $u_n(x,t) = f_n(x)g_n(t)$.

   (f) Solve the *initial value problem*, i.e., find the particular solution $u(x,t)$ that additionally satisfies $u(x,0) = 8\sin x + 11\sin 2x + 15\sin 4x$.

   (g) What is the long-term behavior of this solution (i.e., what happens as $t \to \infty$)?

2. In this problem, we will find the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies the following conditions:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = x(\pi - x), \qquad u_t(x,0) = 0.$$

   Steps (a)–(e) are the same as in the previous problem, and need not be repeated. Instead, repeat part (f) with these new initial conditions. Sketch this scenerio at time $t = 0$.

3. In this problem, we will find the function $u(x,t)$ defined for $0 \leq x \leq \pi$ and $t \geq 0$ which satisfies different initial conditions:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(\pi,t) = 0,$$
$$u(x,0) = 0, \qquad u_t(x,0) = x(\pi - x).$$

   Steps (a)–(c) are the same as in previous problems, and need not be repeated. Instead, repeat parts (d), (e), and (f) with these new initial conditions. What physical situation does this model? Give a physical interpretation for both boundary conditions, and both initial conditions, and sketch this scenerio at time $t = 0$.

4. Which of the following functions are harmonic?
   (a) $f(x) = 10 - 3x$.
   (b) $f(x,y) = x^2 + y^2$.
   (c) $f(x,y) = x^2 - y^2$.
   (d) $f(x,y) = e^x \cos y$.
   (e) $f(x,y) = x^3 - 3xy^2$.

5. (a) Solve the following Dirichlet problem for Laplace's equation in a square region: Find $u(x,y)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0,y) = u(\pi,y) = 0,$$
$$u(x,0) = 0, \quad u(x,\pi) = 4\sin x - 3\sin 2x + 2\sin 3x.$$

   (b) Solve the following Dirichlet problem for Laplace's equation in the same square region: Find $u(x,y)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0,y) = 0, \quad u(\pi,y) = y(\pi - y),$$
$$u(x,0) = u(x,\pi) = 0$$

   (c) By adding the solutions to parts (a) and (b) together (superposition), find the solution to the Dirichlet problem: Find $u(x,y)$, $0 \le x \le \pi$, $0 \le y \le \pi$ such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u(0,y) = 0, \quad u(\pi,y) = y(\pi - y),$$
$$u(x,0) = 0, \quad u(x,\pi) = 4\sin x - 3\sin 2x + 2\sin 3x.$$

   (d) Sketch the solutions to (a), (b), and (c). *Hint: it is enough to sketch the boundaries, and then use the fact that the solutions are harmonic functions.*

   (e) Consider the heat equation in a square region, along with the following boundary conditions:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \qquad u(0,y) = 0, \quad u(\pi,y) = y(\pi - y),$$
$$u(x,0) = 0, \quad u(x,\pi) = 4\sin x - 3\sin 2x + 2\sin 3x.$$

   What is the steady-state solution? (Note: This will *not* depend on the initial conditions!)

6. Consider the following initial/boundary value problem for the heat equation in a square region, and the function $u(x,y,t)$, where $0 \le x \le \pi$, $0 \le y \le \pi$ and $t \ge 0$.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$
$$u(x,0,t) = u(x,\pi,t) = u(0,y,t) = u(\pi,y,t) = 0$$
$$u(x,y,0) = 2\sin x \sin y + 5\sin 2x \sin y.$$

(a) Briefly describe, and sketch, a physical situation which this models. Be sure to explain the effect of the boundary conditions and the initial condition.

(b) Assume that the solution has the form $u(x, y, t) = f(x, y)g(t)$. Find u_{xx}, u_{yy}, and u_t.

(c) Plug $u = fg$ back into the PDE and divide both sides by fg (i.e., "separate variables") to get the *eigenvalue problem.* Briefly justify why this quantity must be a constant. Call this constant λ. Write down an ODE for $g(t)$, and a PDE for $f(x, y)$ (the *Helmholz equation*). Include four boundary conditions for $f(x, y)$.

(d) Solve the Helmholz equation and determine λ. You may assume that $f(x, y) = X(x)Y(y)$.

(e) Solve the ODE for $g(t)$.

(f) Find the general solution of the boundary value problem. It will be a superposition (infinite sum) of solutions $u_{nm}(x, y, t) = f_{nm}(x, y)g_{nm}(t)$.

(g) Find the particular solution to the initial value problem that additionally satisfies the initial condition $u(x, y, 0) = 2 \sin x \sin y + 5 \sin 2x \sin y$.

(h) What is the steady-state solution? Give a mathematical *and* intuitive (physical) justification.

7. Consider the following initial/boundary value problem for the heat equation in a square region, and the funtion $u(x, y, t)$, where $0 \le x \le \pi$, $0 \le y \le \pi$ and $t \ge 0$.

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \\
&u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0 \\
&u(x, y, 0) = (7 \sin x)\, y(\pi - y).
\end{aligned}$$

Since the only difference between this problem and the previous one is in the initial condition, steps (b)–(f) are the same and need not be repeated. Briefly describe, and sketch, a physical situation which this models, and then carry out steps (g) and (h), given this new initial condition.

8. Consider the 2D heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

subject to the boundary conditions

$$\begin{aligned}
&u(x, 0, t) = u(0, y, t) = u(\pi, y, t) = 0 \\
&u(x, \pi, t) = x(\pi - x).
\end{aligned}$$

(a) What is the steady-state solution, $u_{ss}(x, y)$? [Hint: Look at a previous problem on Laplace's equation]. Sketch it.

(b) Write down the general solution this this boundary value problem by adding $u_{ss}(x, y)$ to the general solution of a related *homogeneous* boundary value problem [Hint: Look at a previous problems on the 2D heat equation].

9. Solve the following initial value problem for a vibrating square membrane: Find $u(x, y, t)$, $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ such that

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \\
&u(x, 0, t) = u(x, \pi, t) = u(0, y, t) = u(\pi, y, t) = 0 \\
&u(x, y, 0) = p(x)q(y), \qquad u_t(x, y, 0) = 0.
\end{aligned}$$

where

$$p(x) = \begin{cases} x, & \text{for } 0 \leq x \leq \pi/2, \\ \pi - x, & \text{for } \pi/2 \leq x \leq \pi, \end{cases} \qquad q(y) = \begin{cases} y, & \text{for } 0 \leq y \leq \pi/2, \\ \pi - y, & \text{for } \pi/2 \leq y \leq \pi. \end{cases}$$

(a) Briefly describe a physical situation which this models. Be sure to explain the effect of the boundary conditions and the initial condition. Sketch the initial displacement, $u(x, y, 0)$.

(b) Assume that the solution has the form $u(x, y, t) = f(x, y)g(t)$. Find u_{xx}, u_{yy}, u_t, and u_{tt}.

(c) Plug $u = fg$ back into the PDE and divide both sides by fg (i.e., "separate variables") to get the *eigenvalue problem*. Briefly justify why this quantity must be a constant. Call this constant λ. Write down an ODE for $g(t)$, and a PDE for $f(x, y)$ (the *Helmholz equation*). Include four boundary conditions for $f(x, y)$ and one for $g(t)$.

(d) You may assume that $\lambda = -(n^2 + m^2)$, and that the solution to the Helmholz equation is $f(x, y) = b_{nm} \sin nx \sin my$. Solve the ODE for $g(t)$, using the initial condition.

(e) Find the general solution of the boundary value problem. It will be a superposition (infinite sum) of solutions $u_{nm}(x, y, t) = f_{nm}(x, y)g_{nm}(t)$.

(f) Find the particular solution to the initial value problem that additionally satisfies the initial condition $u(x, y, 0) = p(x)q(y)$.

(g) What is the long-term behavior of $u(x, y, t)$, i.e., as $t \to \infty$. Give a mathematical *and* intuitive (physical) justification.