2. First order differential equations

The order of an ODE is the highest derivative that appears.

Solving ODEs: Like integration, sometimes there's a method, and other times, it's an "art."

Example: Find all solutions to $y' = ky \Rightarrow \frac{dy}{dt} = ky$.

"Magic trick": Multiply thru by dt: $dy = ky\,dt$

Divide thru by y & integrate: $\int \frac{1}{y}\,dy = \int k\,dt$

$$\ln y = kt + c$$

Take exponential of both sides: $y = e^{kt+c}$

$$= e^c e^{kt}$$

let $C = e^c$: $\boxed{y(t) = Ce^{kt}}$

Q: What is C?

A: $y(0)$. "initial condition"

This technique is called separation of variables

Example: (Exponential decay) $y' = -ky$ $(k>0)$.

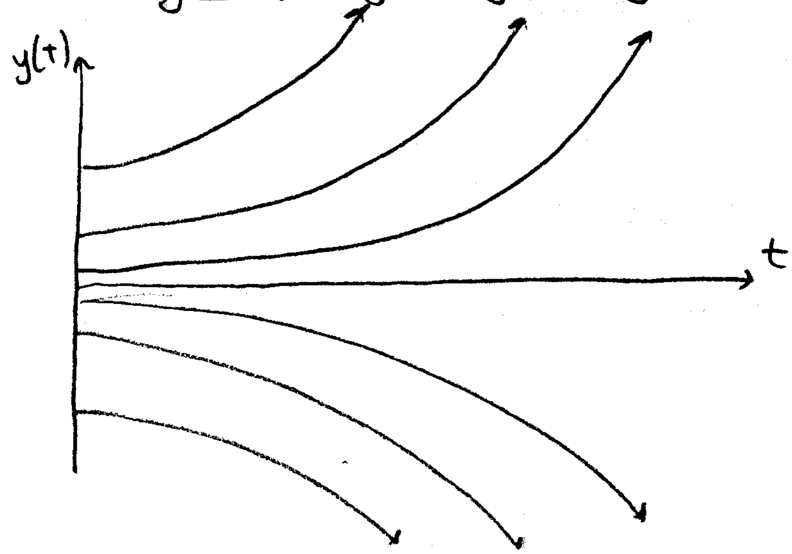
$$\frac{dy}{dt} = -ky \Rightarrow \int \frac{dy}{dt} = \int -k\,dt \Rightarrow \ln y = -kt + c$$

$$\Rightarrow y(t) = e^{-kt+c} = Ce^{-kt}$$

Let's plot the solution (say $k = \frac{1}{10}$).

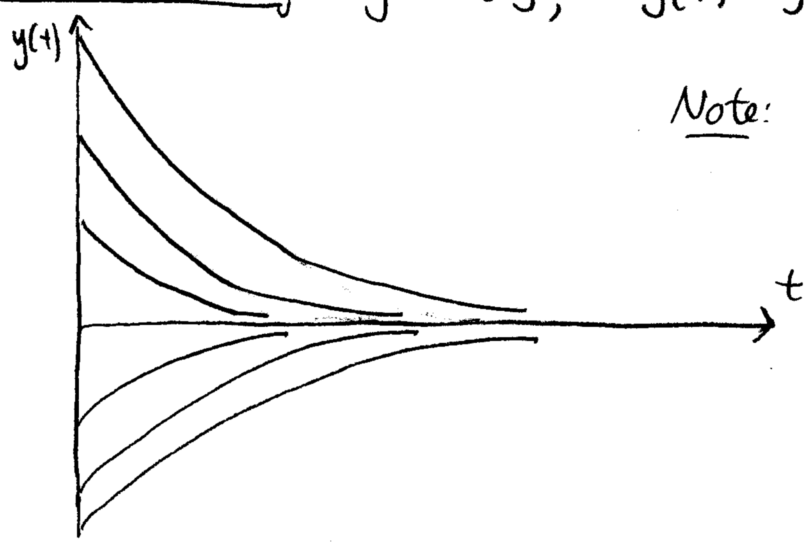
Note: $y' = -ky$ is autonomous; we already know how to do this!

<u>Exponential growth</u>: $y' = \frac{1}{10}y$, $\quad y(t) = y_0 e^{\frac{1}{10}t}$

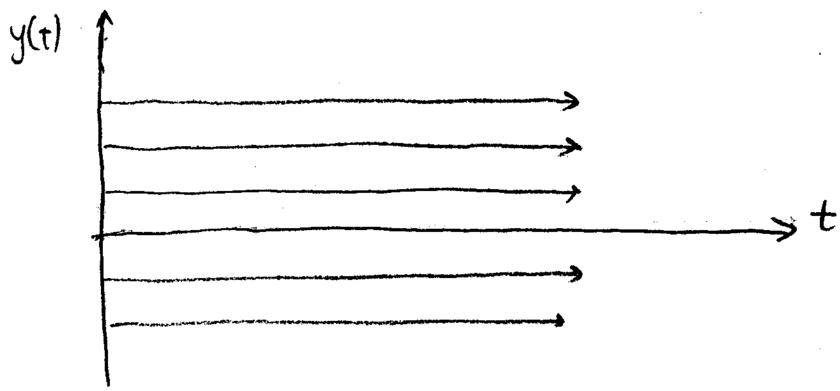


<u>Exponential decay</u>: $y' = -\frac{1}{10}y$, $\quad y(t) = y_0 e^{-\frac{1}{10}t}$

<u>Note</u>: $\lim_{t \to \infty} y(t) = 0.$



<u>What if $k=0$</u>? $\quad y' = 0$, $\quad \frac{dy}{dt} = 0 \Rightarrow y(t) = C$



Q: Can 2 of these <u>solution curves</u> ever intersect?

<u>A</u>: No (why?)

Back to solving ODEs...

<u>Example</u>: (Decay to a limiting value)

$$y' = k(72-y)$$

$$\frac{dy}{dt} = -k(y-72)$$

$$\int \frac{dy}{y-72} = \int k\,dt$$

$$\ln|y-72| = kt + C$$

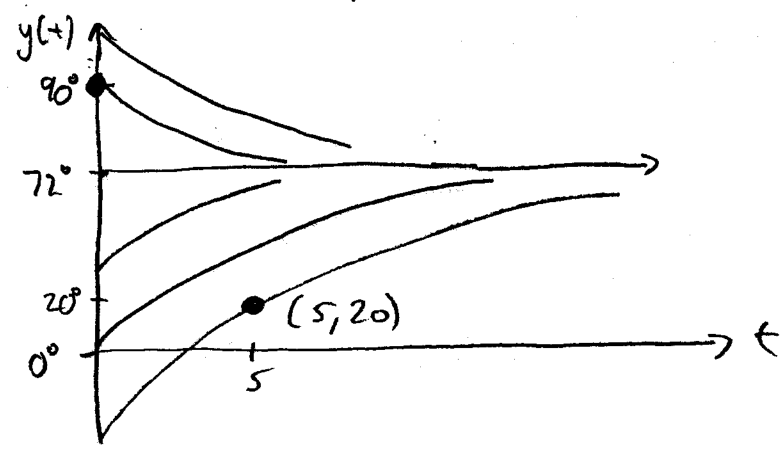
$$y - 72 = e^{kt+c}$$

$$\boxed{y(t) = 72 + Ce^{-kt}}$$

<u>Question</u>: what is "C"?

<u>Ans</u>: $y(0) = 72 + C$

"initial temp. <u>difference</u>"



Initial value problems (IVP's)

- Solving an ODE yields an infinite family of solutions, called the <u>general solution</u>.
- Once we specify a point $(t_0, y(t_0))$, we completely determine a <u>particular solution</u>.

<u>Def</u>: An ODE with a specified point $y(t_0) = y_0$ is called an <u>initial value problem</u> (IVP)

<u>Example</u>: Solve $y' = k(72-y)$, $y(0) = 90$ (see previous example)

$y(t) = 72 + Ce^{-kt}$, $\quad y(0) = 72 + C = 90 \Rightarrow C = 18.$

$\Rightarrow y(t) = 72 + 18e^{-kt}$. * This solution goes thru $(0, 90)$

Suppose instead that $y(5) = 20$. This solution goes thru $(5, 20)$. (See prev. plot)

Solving word problems

Example: (Exponential growth): A house sells in 2003 for \$179,500 and was on sale in 2008 for \$319,500.

(a) What was the average rate of appreciation of the value?

$P'(t) = r P(t)$. Sol'n: $P(t) = Ce^{rt}$.

$P(0) = C = 179,500 \Rightarrow P(t) = 179,500 e^{rt}$

Solve for r: $P(5) = 179,500\, e^{5r} = 319,500$

$$e^{5r} = \frac{3195}{1795} \Rightarrow \ln(e^{5r}) = \ln\left(\frac{3195}{1795}\right)$$

$$\Rightarrow 5r = \ln\frac{3195}{1795} \Rightarrow r = \frac{1}{5}\ln\frac{3195}{1795} \approx 0.115 \quad (11.5\%)$$

(b) Suppose the market has been increasing at a 9% rate. How much is the house worth?

$$r = \frac{9}{100}, \text{ so } P(5) = 179,500\, e^{5\left(\frac{9}{100}\right)} = \$281,512.$$

<u>Example</u>: (Exponential decay). You have 10 grams of a radioactive substance. 3 years later, you have 4 grams.

(a) What is the half-life?

(b) How long until only 1 gram remains?

<u>IVP</u>: $m'(t) = -k\, m(t) \quad m(0) = 10, \quad m(3) = 4.$

<u>General sol'n</u>: $m(t) = Ce^{-kt}$

$m(0) = 10 = C \implies \boxed{m(t) = 10e^{-kt}}$

(a) Half-life is amt. of <u>time</u> until 5 grams remain.

$m(t) = 10e^{-kt} = 5 \implies e^{-kt} = \frac{1}{2} \implies -kt = \ln\frac{1}{2} = -\ln 2$

Solving for t: $\boxed{t_{1/2} = \frac{1}{k}\ln 2}$, thus it only depends on $\underline{\underline{k}}$.

Solve for k: $m(3) = 4 \implies m(3) = 10e^{-3k} = 4$

$\implies e^{-3k} = \frac{4}{10} \implies -3k = \ln\frac{2}{5}$

$\implies k = -\frac{1}{3}\ln\frac{2}{5} \implies \boxed{k = \frac{1}{3}\ln\frac{5}{2}}$

Thus, half-life is $t_{1/2} = \frac{1}{k}\ln 2 = \frac{1}{1/3\,\ln 5/2}\cdot\ln 2 = \boxed{\frac{3\ln 2}{\ln 5/2}}$

(b) How long until 1 gram remains?

$m(t) = 10e^{-kt} = 1$ (solve for t)

$e^{-kt} = \frac{1}{10} \implies -kt = \ln\frac{1}{10} = -\ln 10$

$\implies t = \frac{1}{k}\ln 10 = \boxed{\frac{3\ln 10}{\ln 5/2}}$

<u>Example</u> (Exponential decay to a value): My coffee is 120° when class starts, and the classroom is 75°. After 30 minutes, the coffee is 100°.

(a) What will the temp. be at the end of class (50 min)?

(b) Suppose it was brewed at 160°. When did I brew it?

$$T'(t) = k(75 - T(t)), \quad T(0) = 120, \quad T(30) = 100$$

Sol'n: $T(t) = 75 + Ce^{-kt}$

$T(0) = 75 + C = 120 \Rightarrow C = 45. \Rightarrow \boxed{T(t) = 75 + 45e^{-kt}}$

Need to find k.

(a) $T(30) = 75 + 45e^{-30k} = 100 \Rightarrow 45e^{-30k} = 25$

$\Rightarrow e^{-30k} = \frac{25}{45} \Rightarrow -30k = \ln\frac{25}{45}$

$\Rightarrow k = -\frac{1}{30}\ln\frac{25}{45} = \boxed{\frac{1}{30}\ln\frac{45}{25}}$

$T(t) = 75 + 45e^{-\frac{1}{30}\ln\frac{45}{25}\cdot t}$ $\quad \left(\frac{45}{25} = \frac{9}{5}\right)$

$\boxed{T(50) = 75 + 45e^{-\frac{5}{3}\ln 9/5}}$ (Temp. at end of class)

(b) When was $T(t) = 160$?

$T(t) = 75 + 45e^{-kt} = 160 \Rightarrow 45e^{-kt} = 85$

$\Rightarrow e^{-kt} = \frac{85}{45} = \frac{17}{9} \Rightarrow -kt = \ln\frac{17}{9} \Rightarrow t = -\frac{1}{k}\ln\frac{17}{9}$

The coffee was brewed at $\boxed{t = \frac{-30\ln 17/9}{\ln 9/5}}$

Newton's 2^{nd} law of motion: $F = ma$

Gravitational acceleration: $a = -g = -9.8\ m/s^2$ (why negative?)

Gravitational force: $F = ma = -mg$ (no air resistance)

Add air resistance: $F = -mg + R(v)$. (Forces add).

A good model for air resistance is $R(v) = -rv$ (why?)

"air resistance is proportional to velocity, in the opposite direction."

Therefore, $F = -mg + R(v) = -mg - rv$

(Recall $v'(t) = a(t)$) $\boxed{v' = -g - \frac{r}{m}v}$

Compare to decay → value equation: $T' = k(A - T) = kA - kT$.

Let's put it back into that form: $v' = -g - \frac{r}{m}v = \frac{r}{m}\left(-\frac{mg}{r} - v\right)$

Here, $k = \frac{r}{m}$ and $A = -\frac{mg}{r}$ = limiting (terminal) velocity.

The solution is thus $T(t) = A + Ce^{-kt}$

$$\boxed{v(t) = \frac{-mg}{r} + Ce^{-\frac{r}{m}t}}$$

Remark: Initial velocity $= v(0) = \frac{-mg}{r} + C$

Terminal velocity $= \lim_{t\to\infty} v(t) = \frac{-mg}{r}$

Example: A 70 kg object falls from rest, and its terminal velocity is -20 m/s.

(a) Find its velocity & distance traveled after 2 seconds

(b) How long does it take to reach 80% of its terminal velocity?

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$$v(t) = -\frac{mg}{r} + Ce^{-\frac{r}{m}t} = -20 + Ce^{-\frac{r}{70}t}.$$

$$-\frac{mg}{r} = -20 = -\frac{70g}{r} \Rightarrow \boxed{r = \frac{7}{2}g}$$

$$\Rightarrow v(t) = -20 + Ce^{-\frac{1}{20}gt}$$

$$v(0) = -20 + C = 0 \Rightarrow C = 20.$$

$$\Rightarrow \boxed{v(t) = -20 + 20e^{-\frac{1}{20}gt}}$$

(a) $\boxed{v(2) = -20 + 20e^{-g/10}}$ Recall that $x(t) = \int v(t)\,dt$

$$\text{Dist} = \int_0^2 v(t)\,dt = \int_0^2 -20 + 20e^{-\frac{1}{20}gt}\,dt$$

$$= -20t\Big|_0^2 + \int_0^2 20e^{-\frac{1}{20}gt}\,dt = -40 + \frac{20e^{-\frac{1}{20}gt}}{-1/20g}\Bigg|_0^2$$

$$\boxed{\text{Dist} = -40 - \frac{400}{g}\left(e^{-g/10} - 1\right)}$$

(b) $v(t) = -20 + 20e^{-\frac{1}{20}gt} = -16 \leftarrow$ 80% of -20

$$20e^{-\frac{1}{20}gt} = 4 \Rightarrow e^{-\frac{1}{20}gt} = \frac{1}{5}$$

$$\Rightarrow -\frac{1}{20}gt = \ln\frac{1}{5} = -\ln 5 \Rightarrow \boxed{t = \frac{20\ln 5}{g}}$$

Linear differential equations

Recall high school algebra:

A linear equation is $f(x) = ax + b$.

In MthSc 208:

A (1^{st} order) linear differential equation is $y' = a(t)\,y + f(t)$.

A (1^{st} order) homogeneous linear diff. eq'n is $y' = a(t)\,y$.

Examples:

$y' = t^2 y + 5$	linear
$y' = t y^2 + 5$	non-linear
$y' = t \sin y$	non-linear
$y' = y \sin t$	linear, homogeneous
$y' = t^3 - 2t^2 + t + 1$	linear

We can solve homogeneous ODEs using separation of variables:

$$\frac{dy}{dt} = a(t)\,y \Rightarrow \int \frac{dy}{y} = \int a(t)\,dt \Rightarrow \ln|y| = \int a(t)\,dt + C$$

$$|y| = e^{\int a(t)\,dt + C} \Rightarrow |y| = e^{c} e^{\int a(t)\,dt} \Rightarrow y(t) = C e^{\int a(t)\,dt}$$

Now that we can solve $\underbrace{y'(t) = a(t)\,y(t)}_{\text{linear, homogeneous}}$, let's solve $\underbrace{y'(t) = a(t)\,y(t) + f(t)}_{\text{linear, inhomogeneous}}$

Method #1: Integrating factor ("product rule in reverse")

Write as $\boxed{y'(t) - a(t)\,y(t) = f(t)}$

* Multiply both sides by $e^{-\int a(t)\,dt}$ "integrating factor"

$$e^{-\int a(t)\,dt} y' - a(t) e^{-\int a(t)\,dt} y = f(t) e^{-\int a(t)\,dt}$$

$$\left(y e^{-\int a(t)\,dt}\right)' = f(t) e^{-\int a(t)\,dt}$$ Now integrate both sides.

$$y(t) e^{-\int a(t)\,dt} = \int f(t) e^{-\int a(t)\,dt}\,dt$$

Solving for $y(t)$ ... $\boxed{y(t) = e^{\int a(t)\,dt} \int f(t) e^{-\int a(t)\,dt}\,dt}$

<u>Example</u>: $y' = 2y + t$ <u>Think</u>: why won't sep. of variables work?

$y' - 2y = t$ Integrating factor: e^{-2t}

$$y' e^{-2t} - 2y e^{-2t} = t e^{-2t}$$

$$\int (y e^{-2t})' = \int t e^{-2t}\,dt$$

$$y e^{-2t} = -\tfrac{1}{2} t e^{-2t} - \tfrac{1}{4} e^{-2t} + C$$

$$\boxed{y(t) = -\tfrac{1}{2} t - \tfrac{1}{4} + C e^{2t}}$$

<u>Let's practice getting the integrating factor</u>:

- $y' \boxed{+4} y = t^2$ int. factor e^{4t}; $\frac{d}{dt} e^{4t} = 4e^{4t}$

  $e^{4t} y' + 4e^{t} y = t^2 e^{4t}$

  $(e^{4t} y)' = t^2 e^{4t}$ now integrate & solve.

- $y' \boxed{+(\sin t)} y = 1$ int. factor $e^{-\cos t}$; $\frac{d}{dt} e^{-\cos t} = \sin t\, e^{\cos t}$

  $e^{-\cos t} y' + \sin t\, e^{-\cos t} y = e^{\cos t}$

  $(e^{-\cos t} y)' = e^{-\cos t}$

- $y' \boxed{-12t^5} y = t^3$ int factor e^{-2t^6}, $\frac{d}{dt} e^{-2t^6} = -12t^5 e^{-2t^6}$

$$e^{-2t^6} y' - 12t^5 e^{-2t^6} y = e^{-2t^6} t^3$$

$$\left(e^{-2t^6} y\right)' = e^{-2t^6} t^3$$

- $y' + \boxed{\frac{1}{t} y} = 1$ int. factor $e^{\ln t} = t$, $\frac{d}{dt} t = 1$.

$$e^{\ln t} y' + \frac{1}{t} e^{\ln t} y = t$$

$$t y' + y = t \Rightarrow (ty)' = t.$$

<u>Method #2</u>: <u>Variation of parameters</u>.

<u>Example</u> (same one) $y' = 2y + t$

<u>Step 1</u>: Solve the "homogeneous part:" $y_h' = 2y_h$

$$y_h(t) = Ce^{2t}$$

<u>Step 2</u>: Assume the general sol'n is $y(t) = v(t)\, y_h(t) = v(t) e^{2t}$.

<u>Step 3</u>: Plug this into the ODE & solve for $v(t)$.

$$(ve^{2t})' = 2(ve^{2t}) + t$$

$$\cancel{2ve^{2t}} + v'e^{2t} = \cancel{2ve^{2t}} + t$$

$$v'e^{2t} = t$$

$$v' = te^{-2t} \Rightarrow \int v'(t)\, dt = \int te^{-2t}\, dt$$

$$\Rightarrow v(t) = -\tfrac{1}{2} te^{-2t} - \tfrac{1}{4} e^{-2t} + C$$

<u>Step 4</u>: Plug back into $y(t) = v(t)\, y_h(t)$.

$$y(t) = \left(-\tfrac{1}{2} te^{-2t} - \tfrac{1}{4} e^{-2t} + C\right) e^{2t} \Rightarrow \boxed{y(t) = -\tfrac{1}{2} t - \tfrac{1}{4} + Ce^{2t}}$$

<u>Remark</u>: Both integrating factor & Variation of parameters "work equally well."

<u>Structure of the solutions to 1st order linear ODEs</u>

* <u>Big idea #1</u>: Suppose a <u>homogeneous</u> ODE $y' - a(t)y = 0$ has solutions $y_1(t)$ and $y_2(t)$. Then $C_1 y_1(t) + C_2 y_2(t)$ is a solution for any constants C_1, C_2.

<u>Proof</u>: Plug $C_1 y_1 + C_2 y_2$ back into $y' - a(t)y = 0$:

$$(C_1 y_1 + C_2 y_2)' - a(t)(C_1 y_1 + C_2 y_2) = (C_1 y_1' - a(t) C_1 y_1) + (C_2 y_2' - a(t) C_2 y_2)$$

$$= C_1 \big(\underbrace{y_1' - a(t) y_1}_{=0}\big) + C_2 \big(\underbrace{y_2' - a(t) y_2}_{=0}\big) = 0 \quad \checkmark$$

* <u>Big idea #2</u>: Consider a <u>linear</u> ODE $y' - a(t)y = f(t)$. If $y_p(t)$ is <u>any</u> particular solution, and $y_h(t)$ is the general solution to the related "homogeneous equation," $y' - a(t)y = 0$ the the general solution is $\boxed{y(t) = y_h(t) + y_p(t)}$.

(Recall that $y_h(t) = Ce^{\int a(t)\,dt}$; by separation of variables.)

<u>Proof</u>: If y is the general solution, and y_p <u>any</u> particular solution. Then

$$\begin{array}{l} y' - a(t)y = f \\ -(y_p' - a(t) y_p = f) \\ \hline (y - y_p)' - a(t)(y - y_p) = 0 \end{array}$$

, i.e., <u>$y - y_p$ solves the homogeneous eq'n</u>!

Thus, $y(t) - y_p(t) = y_h(t) = Ce^{\int a(t)\,dt}$, i.e, for *any* $y_p(t)$ that solves the original ODE, we can write the general solution as $y(t) = y_h(t) + y_p(t)$.

Application:
- Solving for $y_h(t)$ is usually easy (separate variables)
- Sometimes, it's easy to see some $y_p(t)$, by inspection.
- When this happens, we automatically have the general solution.

Example: Solve $T' = k(72 - T)$ (quickly!)

Homog. eq'n: $T_h' = -kT$ has sol'n $T_h(t) = Ce^{-kt}$

Find *any* particular sol'n: $T_p(t) = 72$ clearly works (why?)

Thus, the general sol'n is $T(t) = T_h(t) + T_p(t) = 72 + Ce^{-kt}$

Recall: If an ODE is autonomous, set $y' = 0$ to find a constant solution, use this for $y_p(t)$.

Question: Could we have guessed a solution to $y' = 2y + t$?

What if we had tried $y_p(t) = at + b$?

Why is this a good guess ??

Plug $y_p = at + b$, $y_p' = a$ back into $y' = 2y + t$.

$$a = 2(at + b) + t$$

Collect terms: $\underline{0}t + \underline{a} = \underline{(2a+1)}t + \underline{2b}$

Equate coefficients: $\begin{cases} 0 = 2a + 1 \\ a = 2b \end{cases} \Rightarrow a = -\frac{1}{2},\ b = -\frac{1}{4}$

Thus, $y_p(t) = -\frac{1}{2}t - \frac{1}{4}$ is a solution!

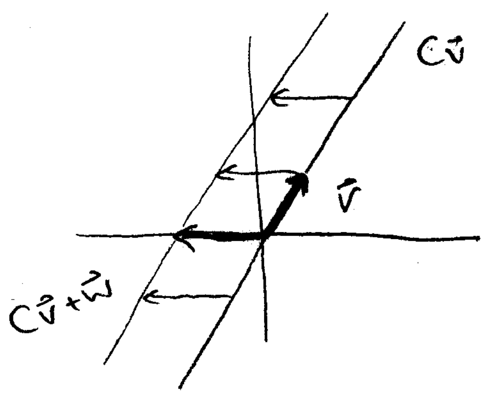
Since $y_h(t) = Ce^{2t}$, the general solution is $y(t) = y_p(t) + y_h(t)$

$$= -\frac{1}{2}t - \frac{1}{4} + Ce^{2t}$$

Think: what does this remind you of?

Recall vector calculus: $\ell = C\vec{v}$ is a line thru $\vec{0}$ $(y = mx)$

(i.e., homogeneous



Q: How do we parametrize a line not thru the origin?

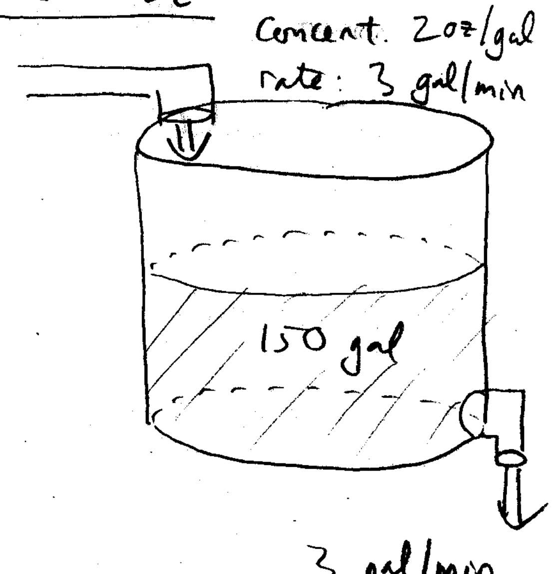
A: Add $\vec{w}$ to it, where $\vec{w}$ is any vector on the line.

All solutions to a linear ODE: $y(t) = C\,y_h(t) + y_p(t)$

All solutions to a linear equation: $\ell = C\,\vec{v} + \vec{w}$ $C \in (-\infty, \infty)$

Some more modeling applications with 1^{st} order ODE's

Mixing problems:



Example 1: Tank of fresh water.

Salt water flows in at some rate

Water drains at some rate.

Q: What is the concentration of salt in the tank at time t?

Let $X(t) =$ # oz salt in the tank at time t.

Note: $\frac{X(t)}{vol(t)} =$ concent. salt at time t.

* Big idea: "$\underbrace{\text{rate of change of salt}}_{x'(t)}$ = rate in − rate out"

rate in = (volume rate)(concentration)

$= \left(3 \frac{gal}{min}\right)\left(2 \frac{oz}{gal}\right) = 6 \ oz/min$

rate out = (volume rate)(concentration)

$= \left(3 \frac{gal}{min}\right)\left(\frac{x(t) \ oz}{150 \ gal}\right) = \frac{1}{50} x(t) \ oz/min.$

Putting this together: $\boxed{x'(t) = 6 - \frac{1}{50} x(t), \quad x(0) = 0}$

Initially contains fresh water

Let's solve this.

We could use (i) Separation of variables
(ii) Integrating factor
(iii) Variation of parameters
(iv) $y(t) = y_h(t) + y_p(t)$.

Let's use (iv); it's easier.

To find a steady-state (constant) solution, set $x_p' = 0$:

$0 = 6 - \frac{1}{50} x_p \implies x_p = 300$

The homogeneous eq'n is $x_h' = -\frac{1}{50} x_h$, has sol'n $x_h(t) = Ce^{-\frac{1}{50}t}$.

Thus, the general solution is $x(t) = x_h(t) + x_p(t)$

$$\boxed{x(t) = Ce^{-\frac{1}{50}t} + 300}$$

Initial condition: $x(0) = 0$ (fresh water $\Rightarrow$ no salt)

$$x(0) = 300 + C = 0 \quad \Rightarrow \quad C = -300$$

$\Rightarrow$ $\boxed{x(t) = 300 - 300e^{-\frac{1}{50}t}}$ is the sol'n to the IVP.

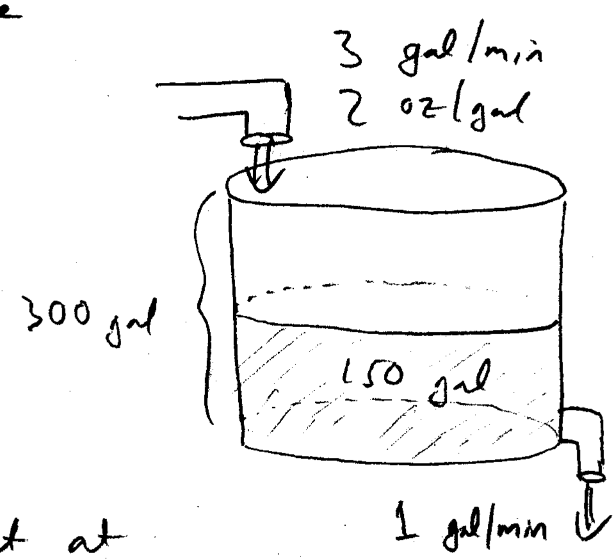
Note: $\lim_{t\to\infty} x(t) = 300$ i.e, the amount of salt $\longrightarrow$ 300 oz

Concentration at time t : $\dfrac{x(t)}{150} = 2 - 2e^{-\frac{1}{50}t}$

Remark: Mathematically, this is the same as:

- Newton's law of cooling
- Falling objects with air resistance

Example 2:



Tank of fresh water
Salt water flows in at a faster rate (3 gal/min) than it drains (1 gal/min)

Q: What is the concentration of salt at the moment it overflows

Note: $\mathrm{vol}(t) = 150 + 2t$, so it overflows at $t = 75$ min.

$$x'(t) = (\text{rate in}) - (\text{rate out})$$

$$\text{rate in} = \left(3 \tfrac{gal}{min}\right)\left(2 \tfrac{oz}{gal}\right) = 6 \tfrac{oz}{min}$$

$$\text{rate out} = \left(1 \tfrac{gal}{min}\right)\left(\frac{x(t)}{150+2t} \frac{oz}{gal}\right) = \frac{1}{150+2t}\, x(t) \frac{oz}{min}.$$

IVP: $\boxed{x' = 6 - \frac{1}{150+2t}x, \quad x(0)=0}$

Let's solve this: $x' + \frac{1}{150+2t}x = 6$ int. factor: $e^{\int \frac{1}{150+2t}dt}$

Note: $\int \frac{1}{150+2t} = \frac{1}{2}\int \frac{1}{75+t}dt = \frac{1}{2}\ln(75+t)+C$

Int factor $= e^{\int \frac{1}{150+2t}dt} = \left(e^{\ln(75+t)}\right)^{1/2} = (75+t)^{1/2} = \sqrt{75+t}$

$$\left(x\cdot(75+t)^{1/2}\right)' = 6(75+t)^{1/2}$$

$$\Rightarrow x\cdot(75+t)^{1/2} = 6\int(75+t)^{1/2}dt = 4(75+t)^{3/2}+C$$

$$\Rightarrow x(t) = 4(75+t) + C(75+t)^{-1/2}$$

$$\Rightarrow \boxed{x(t) = 300+4t+\frac{C}{\sqrt{75+t}}}$$

Use $x(0)=0$: $x(0) = 300 + \frac{C}{\sqrt{75}} = 0 \Rightarrow C = -300\sqrt{75}$

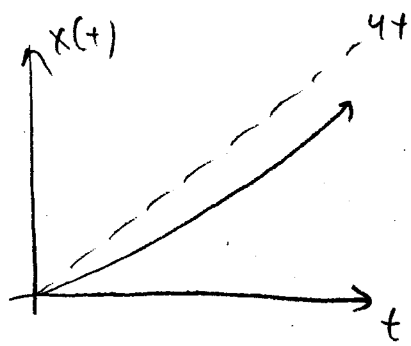
Final solution: $\boxed{x(t) = 300+4t-\frac{300\sqrt{75}}{\sqrt{75+t}}}$

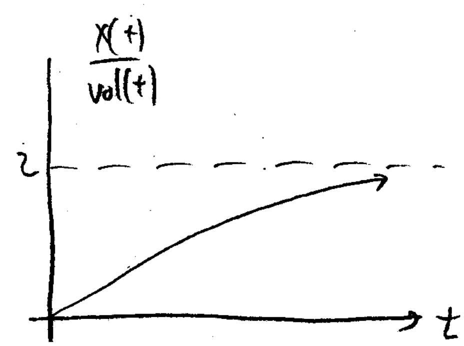
At $t=75$, when the tank overflows, the salt content is

$x(75) = 600 - \frac{300}{\sqrt{2}} \approx 387.87$ oz, so the concentration is

$\frac{x(75)\text{ oz}}{300\text{ gal}} \approx \frac{387.87}{300} = \boxed{1.29\text{ oz/gal}}$

This makes sense:





<u>Question</u>: Could we have <u>guessed</u> a particular sol'n to $x' = 6 - \frac{1}{150+2t}x$?

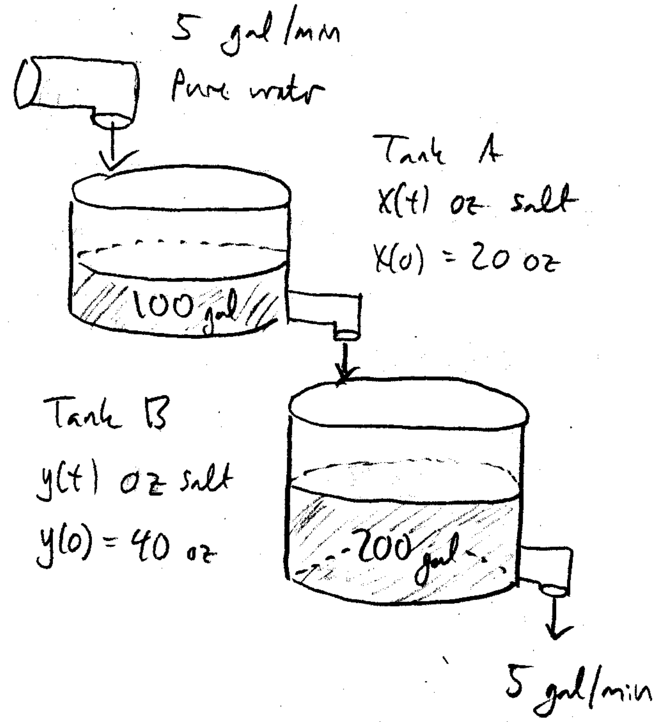
Consider the <u>simplest</u> scenerio: When $x(0) = 300$.

In this case the concentration is $2\ ^{oz}/_{gal}$ for all t.

Thus $x(t) = (2\ ^{oz}/_{gal})\ \text{Vol}(t) = 2(150+2t) = 300+4t$.

We could have used this for $x_p(t)$, then solved the homogeneous eq'n, $x_h' = -\frac{1}{150+2t}x_h$!

<u>Example 3</u>: Mixing with 2 tanks.



Let $x(t)$ = # oz salt in tank A

$y(t)$ = # oz salt in tank B.

<u>Tank A</u>: $x'(t) = (\text{rate in}) - (\text{rate out})$

rate in: $\left(5\ \frac{gal}{min}\right)\left(0\ \frac{oz}{gal}\right) = 0$

rate out: $\left(5\ \frac{gal}{min}\right)\left(\frac{x(t)\ oz}{100\ gal}\right) = \frac{1}{20}x$

<u>Tank B</u>: $y'(t) = (\text{rate in}) - (\text{rate out})$

rate in $= \left(5\ \frac{gal}{min}\right)\left(\frac{x(t)\ oz}{100\ gal}\right) = \frac{1}{20}x$

rate out $= \left(5\ \frac{gal}{min}\right)\left(\frac{y(t)\ oz}{200\ gal}\right) = \frac{1}{40}y$

We get a <u>system</u> of ODE's:

$$\boxed{\begin{array}{ll} x' = -\frac{1}{20}x & x(0) = 20 \\ y' = \frac{1}{20}x - \frac{1}{40}y & y(0) = 40 \end{array}}$$

$x(t) = 20e^{-\frac{1}{20}t}$ (why?). Plug this into the 2nd ODE.

$$y' = \frac{1}{20}\left(20 e^{-\frac{1}{20}t}\right) - \frac{1}{40} y$$

$$y' + \frac{1}{40} y = e^{-\frac{1}{20}t} \qquad \text{int. factor} = e^{\frac{1}{40}t}$$

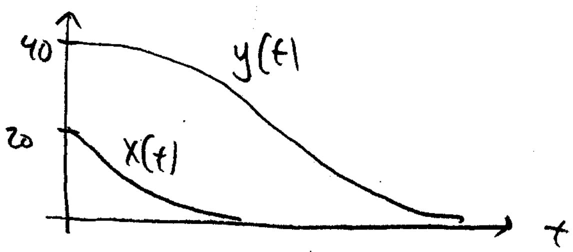
$$\left(y e^{\frac{1}{40}t}\right)' = e^{-\frac{1}{20}t} \cdot e^{\frac{1}{40}t} = e^{-\frac{1}{40}t}$$

$$y e^{\frac{1}{40}t} = \int e^{-\frac{1}{40}t}\,dt = -40 e^{-\frac{1}{40}t} + C$$

$$y(t) = \left(-40 e^{-\frac{1}{40}t} + C\right) e^{-\frac{1}{40}t} = -40 e^{-\frac{1}{20}t} + C e^{-\frac{1}{40}t}$$

$$y(0) = -40 + C = 40 \implies C = 80$$

$$y(t) = -40 e^{-\frac{1}{20}t} + 80 e^{-\frac{1}{40}t}$$



Logistic equation

Recall exponential growth: $y'(t) = r\, y(t)$, rate r does not depend on $y(t)$.

Suppose $y(t)$ = population of a colony.

- When $y(t)$ is small, it grows exponentially.
- When $y(t)$ is large, it grows slowly (decays → "capacity")
- When $y(t) >$ capacity, it decreases.

* In general, the "rate" r decreases as $y(t)$ increases.

How do we model this?

We want $y'(t) = r(y)\, y(t)$, where $r(y)$ is decreasing.

Try $r(y) = r - ay$, where $a > 0$ fixed constant (it's simple!)

<u>Check</u>: When $y=0$, $r(y)=r$ ✓

When $y=\frac{r}{a}$, $r(y)=0$ ✓

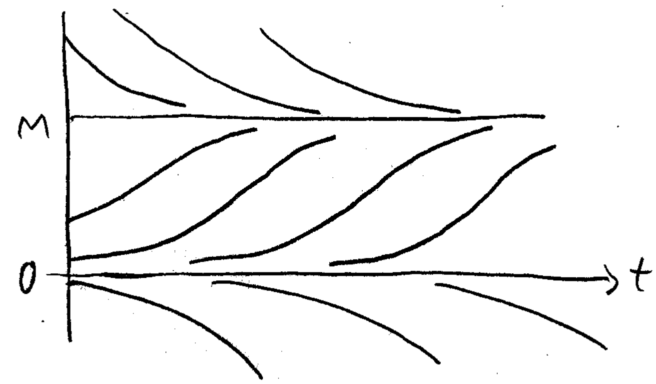
When $y>\frac{r}{a}$, $r(y)<0$ ✓

We call this threshold $M := \frac{r}{a}$ the <u>carrying capacity</u>; $a=\frac{r}{M}$

We have $y'(t) = r(y)\,y(t) = (r-ay)y = \left(r-\frac{r}{M}\right)y = ry\left(1-\frac{y}{M}\right)$

* The equation $\boxed{y' = ry\left(1-\frac{y}{M}\right)}$ is the <u>logistic equation</u>.

The steady-state solutions are $y(t)=0$, $y(t)=M$.



We can solve this by separation of variables (it's messy).

The solution is $\boxed{y(t) = \frac{M}{1+Ce^{-rt}}}$

<u>Remark</u>: Initial population: $y(0) = \frac{M}{1+C}$

Limiting population: $\lim_{t\to\infty} y(t) = M$.

<u>Example</u>: The mass of a colony of bacteria satisfies the logistic equation. The petrie dish holds 50 grams. Initially, there are 10 grams & mass is increasing at 1 gram/day. Find $m(t)$.

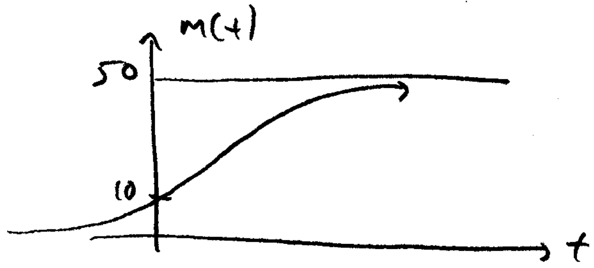
We know $M=50 \Rightarrow m(t) = \frac{50}{1+Ce^{-rt}}$, $m(0)=10$, $m'(0)=1$

$m(0) = \frac{50}{1+C} = 10 \Rightarrow C=4 \Rightarrow m(t) = \frac{50}{1+4e^{-rt}}$

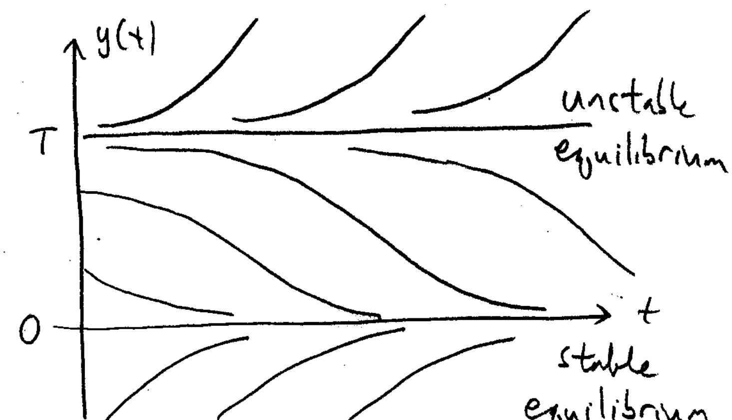
$$m'(0) = r\left(1 - \frac{m(0)}{50}\right) m(0)$$

$$1 = r\left(1 - \frac{10}{50}\right)\cdot 10 = r \cdot \frac{4}{5} \cdot 10 = 1 \implies r = \frac{1}{8}.$$

The particular sol'n is thus $\boxed{m(t) = \dfrac{50}{1 + 4e^{-t/8}}}$



<u>Question</u>: What if we replace r with $-r$ in the logistic equation?



We'd get an ODE (let $T = M$)

$$\boxed{y' = -r\left(1 - \frac{y}{T}\right) y}$$

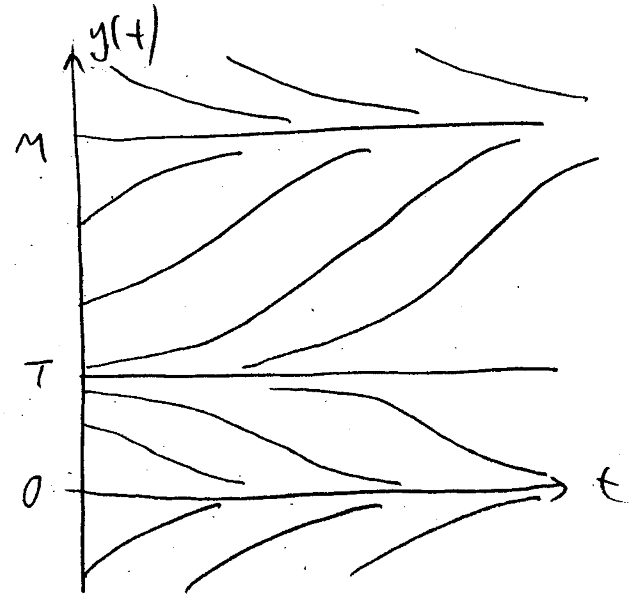
This models a population with an "extinction threshold," i.e.,

- If $y(t) < T$, population dies out, but
- If $y(t) > T$, population "explodes."

Realistically, we'd like a model that captures <u>both</u> phenomena.

Let's <u>make</u> an ODE with steady-state solutions $y(t) = 0, M, T$.

$$y'(t) = -ry\left(1 - \frac{y}{M}\right)\left(1 - \frac{y}{T}\right).$$



* This actually modeled the (now extinct) passenger pigeon quite well!