6. Fourier Series:

* Every "well-behaved" periodic function (think: arbitrary sound wave) can be decomposed into sine & cosine waves.

We'll learn how to do this. It will be necessary for the study of partial differential equations. (e.g., $\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$).

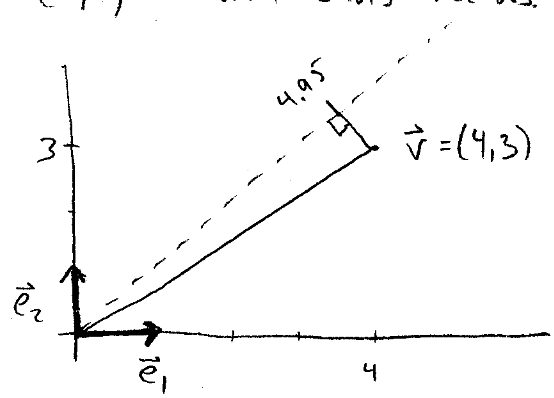
Motivation: $\mathbb{R}^n$ is a set of vectors

We can add & subtract vectors, and we know how to "measure" their lengths: $\|\vec{v}\| := \sqrt{\vec{v}\cdot\vec{v}}$.

e.g., $\|(4,3)\| = \sqrt{4^2+3^2} = 5$.

We can also project a vector onto a unit vector using the dot product.

Example: Let $\vec{v} = (4,3)$ and let $\vec{e}_1 = (1,0)$, $\vec{e}_2 = (0,1)$ "unit basis vectors."



Q: How long is $\vec{v}$ in the x-direction?

A: $\vec{v}\cdot\vec{e}_1 = (4,3)\cdot(1,0) = 4$.

Q: How long is $\vec{v}$ in the y-direction?

A: $\vec{v}\cdot\vec{e}_2 = (4,3)\cdot(0,1) = 3$

Q: How long is $\vec{v}$ in the "northeast" or (1,1)-direction?

A: $\vec{v}\cdot\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) = (4,3)\cdot\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) = \frac{7\sqrt{2}}{2} \approx 4.95$.

unit vector in the "(1,1)-direction"

The unit basis vectors $\{\vec{e}_1, \dots, \vec{e}_n\}$ of $\mathbb{R}^n$ have some nice properties:

(i) $\|\vec{e}_i\| = \sqrt{\vec{e}_i \cdot \vec{e}_i} = 1$ ($\vec{e}_i$ has length 1)

(ii) If $i \neq j$, then $\vec{e}_i \cdot \vec{e}_j = 0$ ($\vec{e}_i$ & $\vec{e}_j$ are <u>orthogonal</u> (perpendicular)).

Together, we can summarize this by $\vec{e}_i \cdot \vec{e}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$.

<u>Def</u>: A set of vectors is orthonormal if they satisfy conditions (i) & (ii) above.

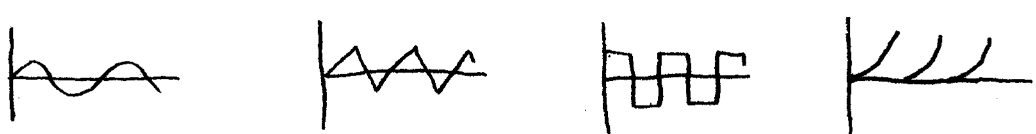
<u>Easy fact</u>: $\{\vec{e}_1, \dots, \vec{e}_n\}$ is an <u>orthonormal basis</u> of $\mathbb{R}^n$

Because of this, we can <u>decompose</u> any vector in $\mathbb{R}^n$ into components, by <u>projecting</u> onto the basis vectors.

e.g., $\vec{v} = (5,4,3) = 5\vec{e}_1 + 4\vec{e}_2 + 3\vec{e}_3 = (\vec{v} \cdot \vec{e}_1)\vec{e}_1 + (\vec{v} \cdot \vec{e}_2)\vec{e}_2 + (\vec{v} \cdot \vec{e}_3)\vec{e}_3$.

This is precisely the technique that we'll use to decompose a periodic function into sine & cosine waves!

* Let $\mathrm{Per}_{2\pi}$ be the set of 2π-periodic piecewise continuous functions.

e.g.,  etc.

We can think of these functions as <u>vectors</u>.

We can add & subtract these "vectors" & multiply them by scalars.

We need to define a "dot product," (called an <u>inner product</u>) so we can measure their lengths.

Define $\boxed{\langle f(x), g(x) \rangle := \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, g(x)\, dx}$

Remark: $\langle f, g\rangle$ is just a preferred notation for "$f \cdot g$"

This defines "length"; $\|\vec{v}\|^2 = \langle \vec{v} \cdot \vec{v}\rangle$ (as in $\mathbb{R}^n$)

so $\|f(x)\|^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx$

* Key fact: The set $\mathcal{B}_{2\pi} := \left\{ \begin{matrix} \frac{1}{\sqrt{2}}, & \cos x, & \cos 2x, & \cos 3x, \ldots \\ & \sin x, & \sin 2x, & \sin 3x, \ldots \end{matrix} \right\}$

is an orthonormal basis for $Per_{2\pi}$, given our definition of length!

i.e., $\langle \cos nx, \cos mx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \cos mx\,dx = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

$\langle \sin nx, \sin mx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx\,dx = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$

$\langle \cos nx, \sin mx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx \sin mx\,dx = 0.$

Now, we automatically know how to decompose a periodic function into sines & cosines – just "project" onto the basis vectors in $\mathcal{B}_{2\pi}$.

Let $f(x)$ be a piecewise continuous 2π-periodic function.

We can write $\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx}$

a_n = "length of $f(x)$ in the $(\cos nx)$-direction"

b_n = "length of $f(x)$ in the $(\sin nx)$-direction"

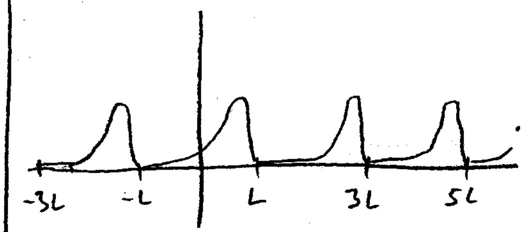
and

$$\boxed{\begin{aligned} a_n &= \langle f(x), \cos nx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx \\ b_n &= \langle f(x), \sin nx\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx \end{aligned}}$$

Note: This formula works for a_0 too: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$.

Remark: This easily generalizes to functions of period $2L$ (not just 2π):

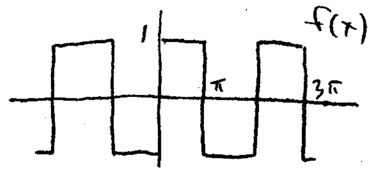
$$\boxed{\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \\ a_n &= \frac{2}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \\ b_n &= \frac{2}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \end{aligned}}$$



[Show demo: www.falstad.com/fourier]

However, the math is messier for $L \neq \pi$, so we'll just stick with 2π-periodic functions in this class.

Example 1: Square wave:



Find the Fourier series of $f(x)$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -1\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\,dx = 0$$

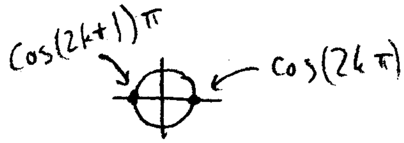
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -1\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\cos nx\,dx$$

$$= -\frac{1}{n\pi}\sin nx\Big|_{-\pi}^{0} + \frac{1}{n\pi}\sin nx\Big|_{0}^{\pi} = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -1\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} 1\sin nx\,dx$$

$$= \frac{1}{n\pi}\cos nx\Big|_{-\pi}^{0} - \frac{1}{n\pi}\cos n\pi\Big|_{0}^{\pi} = \frac{1}{n\pi}(1-\cos n\pi) - \frac{1}{n\pi}(\cos n\pi - 1)$$

$$= \boxed{\frac{2}{n\pi}(1-\cos n\pi)}$$

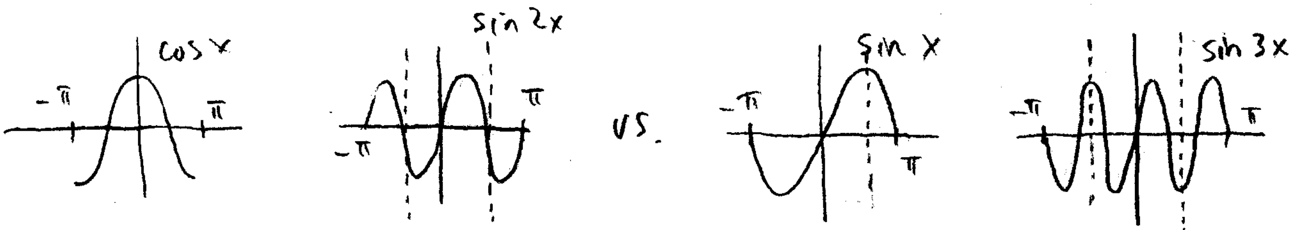


Note: $\cos n\pi = (-1)^n$

Therefore, $b_n = \frac{2}{n\pi}\left(1-(-1)^n\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases}$

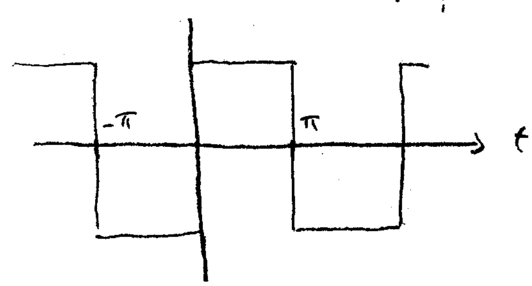
i.e., $\boxed{f(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \frac{4}{7\pi}\sin 7x + \ldots}$

Note: All cosine terms, and "even-index" sine terms are zero. (why?)

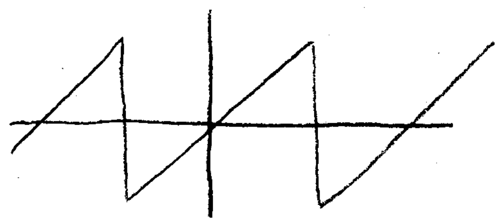


Look at the "symmetries" of $f(x)$:

This "looks" like a sine wave,
and "more like" a $\sin x$, $\sin 3x$, etc. than a $\sin 2x$, $\sin 4x$, etc. function.



Example 2: Sawtooth wave:



$f(x) = x$ on $[-\pi, \pi]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx = 0 \quad \text{(By symmetry; area under the curve)}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx\,dx \qquad \begin{aligned} \text{let } u &= x & v &= \tfrac{1}{n}\sin nx \\ du &= dx & dv &= \cos nx\,dx \end{aligned}$$

$$= \frac{1}{\pi}\left[\frac{1}{n} x \overbrace{\cancel{\sin} nx}^{0}\Big|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi} \sin nx\,dx\right]$$

$$= -\frac{1}{n\pi}\int_{-\pi}^{\pi} \sin nx\,dx = \frac{1}{n^2\pi}\cos nx\Big|_{-\pi}^{\pi} = \frac{1}{n^2\pi}\left[\cos(\pi x) - \cos(-\pi x)\right]$$

$$= \frac{1}{n^2\pi}\left[\cos \pi x - \cos n\pi\right] = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin nx \, dx \qquad \text{let } u = x \quad v = -\frac{1}{n}\cos nx \qquad du = dx \quad dv = \sin nx \, dx$$

$$= \frac{1}{\pi}\left[-\frac{1}{n} x \cos nx \Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} \cos nx \, dx \right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi}{n}\cos n\pi\right) - \left(\frac{\pi}{n}\cos n\pi\right) + \frac{1}{n^2}\cancelto{0}{\sin nx}\Big|_{-\pi}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{n}\cos(n\pi)\right] = -\frac{2}{n}\cos n\pi = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1} = \begin{cases} -2/n & n \text{ even} \\ 2/n & n \text{ odd} \end{cases}$$

Thus, $f(x) = 2\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \frac{2}{5}\sin 5x \pm \ldots$

$$= 2\sin x - \sin 2x + \frac{2}{3}\sin 3x - \frac{1}{2}\sin 4x + \frac{2}{5}\sin 5x \pm \ldots$$

Think: How does this relate to music, sound waves, & harmonics?

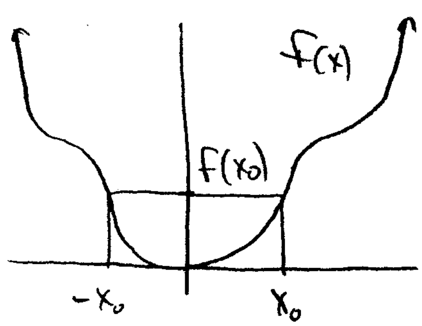
Exploiting symmetry:

Why are many of the a_n's & b_n's zero?

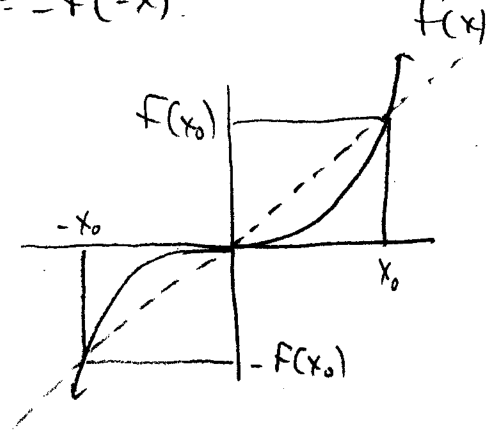
Def: • $f(x)$ is an even function if $f(x) = f(-x)$

• $f(x)$ is an odd function if $f(x) = -f(-x)$.

Graphically,



$f(x)$ even $\Leftrightarrow$ symmetric about the y-axis.



$f(x)$ odd $\Leftrightarrow$ symmetric about the origin.

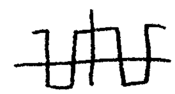
Why we care:

- If $f(x)$ is even, then $\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$
- If $f(x)$ is odd, then $\int_{-L}^{L} f(x)\,dx = 0$.

Look at the area under the curve to see why!

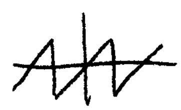
Basic facts:
- If f & g are even, then $f(x)\,g(x)$ is even.
- If f & g are odd, then $f(x)\,g(x)$ is even.
- If f is even & g is odd, then $f(x)\,g(x)$ is odd.

Examples:

- Even functions: 8, x^2, $3x^6 + x^2 - 5$, $|x|$, 

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} +/- \cdots = \frac{e^{ix} + e^{-ix}}{2}$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \frac{e^{x} + e^{-x}}{2}$$

- Odd functions: $2x$, $8x^3 - 5x$,  

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} +/- \cdots = \frac{e^{ix} - e^{-ix}}{2i}$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \frac{e^{x} - e^{-x}}{2}$$

- Neither: $x^2 - 3x + 2$, $x^5 + x^3 + x + 1$, $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Off-hand remark:

* $\cos x = \cosh ix$
* $i \sin x = \sinh ix$
* $e^x = \cosh x + \sinh x = \cos x + i \sin x$

Key point:

- If $f(x)$ is <u>even</u>, then $f(x)\cos nx$ is <u>even</u> $\Rightarrow a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$

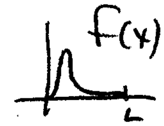
  and $f(x)\sin nx$ is <u>odd</u> $\Rightarrow b_n = 0$ (all n)

- If $f(x)$ is <u>odd</u>, then $f(x)\cos nx$ is <u>odd</u> $\Rightarrow a_n = 0$ (all n)

  and $f(x)\sin nx$ is <u>even</u> $\Rightarrow b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$

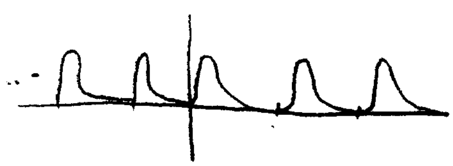
Fourier sine & cosine series

<u>Idea</u>: Consider a function defined on $[0,L]$

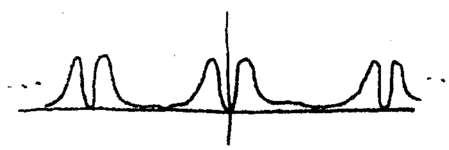


Write $f(x)$ as a Fourier series.

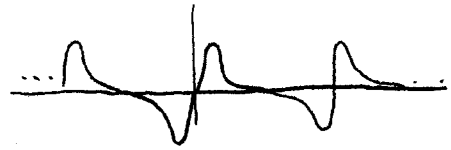
First, we need to <u>make</u> $f(x)$ periodic.



A naive extension



The <u>even</u> extension



The <u>odd</u> extension

<u>Def</u>: The <u>Fourier cosine series</u> of $f(x)$ is the Fourier series of the <u>even</u> extension of $f(x)$

$$\begin{cases} a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n = 0 \end{cases}$$

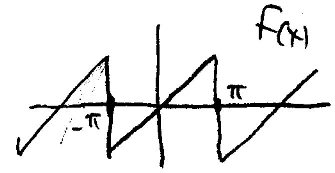
<u>Def</u>: The <u>Fourier sine series</u> of $f(x)$ is the Fourier series of the <u>odd</u> extension of $f(x)$

$$\begin{cases} a_n = 0 \\ b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{cases}$$

Example 3: Let $f(x) = x$ on $[0, \pi]$

Compute the Fourier sine & cosine series of $f(x)$.

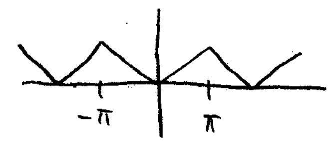
Fourier sine series: Odd extension:



This was Example 2, on p. 5-6.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \qquad b_n = \begin{cases} -2/n\pi & n \text{ even} \\ 2/n\pi & n \text{ odd} \end{cases}$$

Fourier cosine series: Even extension:



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{x^2}{\pi}\Big|_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx = \frac{2}{\pi}\left[\cancelto{0}{\frac{x}{n}\sin nx\Big|_0^{\pi}} - \int_0^{\pi}\frac{1}{n}\sin nx\,dx\right]$$

Let $u = x$, $v = \frac{1}{n}\sin nx$; $du = dx$, $dv = \cos nx\,dx$

$$= \frac{2}{\pi n^2}\cos nx\Big|_0^{\pi} = \frac{2}{n^2\pi}\left[\cos n\pi - 1\right]$$

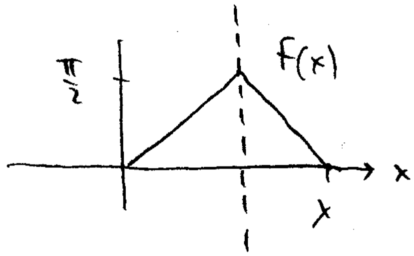
$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right] = \begin{cases} 0 & n \text{ even} \\ \dfrac{-4}{\pi n^2} & n \text{ odd} \end{cases}$$

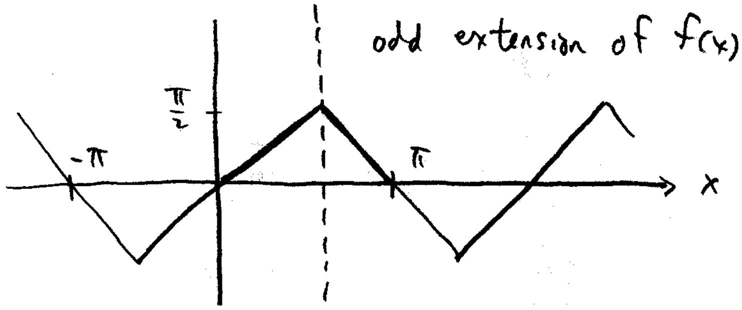
Thus, $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x - \frac{4}{25\pi}\cos 5x - \frac{4}{49\pi}\cos 7x - \cdots$

Example 4: Let $f(x) = \begin{cases} x & 0 \le x < \pi/2 \\ \pi - x & \pi/2 \le x < \pi \end{cases}$

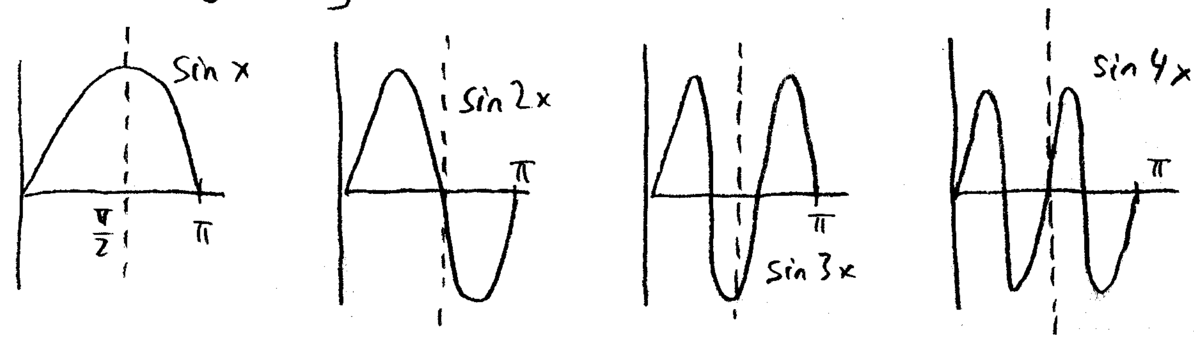
Compute the Fourier sine series of $f(x)$.

$\boxed{10}$





Observe the symmetry about the line $x = \pi/2$.



* $\sin nx$ has odd symmetry about the line $x = \pi/2$ if n is even
* $\sin nx$ has even symmetry about the line $x = \pi/2$ if n is odd.

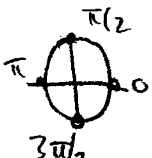
Conclusion: If n is even, then $b_n = 0$.

If n is odd, then $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{4}{\pi}\int_0^{\pi/2} f(x)\sin nx\,dx$

$$= \frac{4}{\pi}\int_0^{\frac{\pi}{2}} x\sin nx\,dx = \frac{4}{\pi}\left[\frac{x}{n}\cos nx\Big|_0^{\frac{\pi}{2}} + \int_0^{\pi/2}\frac{1}{n}\cos nx\,dx\right]$$

$$= \frac{4}{\pi}\left[\frac{-\pi}{2\pi}\cancelto{0}{\cos\left(\frac{n\pi}{2}\right)} - 0 + \frac{1}{n^2}\sin nx\Big|_0^{\pi/2}\right] \qquad n \text{ odd} \Rightarrow \cos\left(\frac{n\pi}{2}\right) = 0$$

$$= \frac{4}{\pi}\left[\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\right]$$



$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & n = 4k \\ 1 & n = 4k+1 \\ 0 & n = 4k+2 \\ -1 & n = 4k+3 \end{cases}$$

$$\text{Thus, } b_n = \begin{cases} 0 & n = 4k \\ 4/n^2\pi & n = 4k+1 \\ 0 & n = 4k+2 \\ -4/n^2\pi & n = 4k+3 \end{cases}$$

So, $f(x) = \frac{4}{\pi}\sin x - \frac{4}{9\pi}\sin 3x + \frac{4}{25\pi}\sin 5x - \frac{4}{49\pi}\sin 7x +/- \cdots$

Complex form of Fourier series

Recall: $\mathcal{B}_1 = \left\{ \begin{matrix} \frac{1}{\sqrt{2}}, & \cos x, & \cos 2x, & \cos 3x, \dots \\ & \sin x, & \sin 2x, & \sin 3x, \dots \end{matrix} \right\}$ is a basis for $Per_{2\pi}$

Fact: $\mathcal{B}_2 = \left\{ \begin{matrix} 1, & e^{ix}, & e^{2ix}, & e^{3ix}, \dots \\ & e^{-ix}, & e^{-2ix}, & e^{-3ix}, \dots \end{matrix} \right\}$ is also a basis for $Per_{2\pi}$:

and is orthonormal if $\langle f, g \rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\overline{g(x)}\,dx$.

Therefore, if $f(x)$ is 2π-periodic, we can write $f(x)$ as

$$\boxed{\begin{gathered} f(x) = \sum_{-\infty}^{\infty} c_n e^{inx} = c_0 + \sum_{n=1}^{\infty} \left(c_n e^{inx} + c_{-n} e^{-inx}\right). \\ c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,e^{-inx}\,dx \end{gathered}}$$

This is the complex form of the Fourier series of $f(x)$.

Recall: $\cos nx = \frac{1}{2}\left(e^{inx} + e^{-inx}\right)$, $\quad \sin nx = \frac{1}{2i}\left(e^{inx} - e^{-inx}\right)$

$e^{inx} = \cos nx + i \sin nx$, $\quad e^{-inx} = \cos nx - i \sin nx$

Therefore, $\boxed{c_n = \frac{a_n - i b_n}{2}, \qquad c_{-n} = \frac{a_n + i b_n}{2}}$

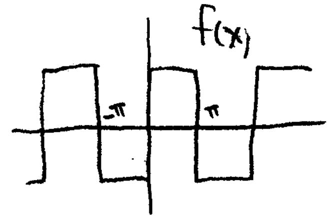
and $\boxed{a_n = c_n + c_{-n} \qquad b_n = i(c_n - c_{-n})}$

Note: c_0 is the constant term in the complex version of the Fourier series.

$a_0 = 2c_0 \Rightarrow \frac{a_0}{2}$ is the const. term in the real version.

Remark: The const term c_0 (or $\frac{a_0}{2}$) is the average value of $f(x)$ (why?)

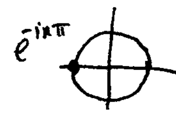
Example 1: Compute the complex Fourier series of



$c_0 = 0$ (average value of $f(x)$).

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{0} -e^{-inx}\,dx + \frac{1}{2\pi}\int_{0}^{\pi} e^{-inx}\,dx$$

$$= \frac{1}{2\pi}\left[\frac{1}{in} e^{-inx}\right]_{-\pi}^{0} + \frac{1}{2\pi}\left[\frac{-1}{in} e^{-inx}\right]_{0}^{\pi}$$



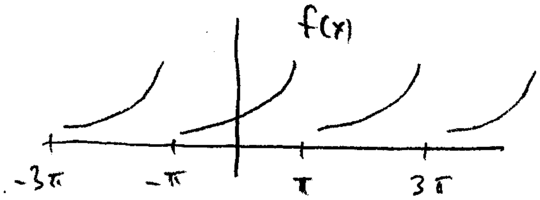
$$= \frac{1}{2\pi i n}\left(1 - e^{in\pi} - e^{-in\pi} + 1\right)$$

Note: $e^{-in\pi} = e^{in\pi} = (-1)^n = (-1)^{-n}$.

$$= \boxed{\frac{1}{\pi i n}\left(1 - (-1)^n\right)} = \begin{cases} \dfrac{2}{\pi i n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Thus, $$\boxed{f(x) = \sum_{\substack{n=-\infty \\ n\neq 0}} \frac{1}{\pi i n}\left(1-(-1)^n\right) e^{inx} = \sum_{n=1}^{\infty} \frac{1}{\pi i n}\left(1-(-1)^n\right)\left(e^{inx} - e^{-inx}\right)}$$

Example 2: Compute the complex Fourier series of the 2π-periodic extension of e^x (defined on $[-\pi, \pi]$).



$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{2\pi} e^x \Big|_{-\pi}^{\pi} = \boxed{\frac{1}{2\pi}\left(e^{\pi} - e^{-\pi}\right)}$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-in)x}\,dx = \frac{1}{2\pi(1-in)} e^{(1-in)x}\Big|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi(1-in)}\left[e^{(1-in)\pi} - e^{-(1-in)\pi}\right] = \frac{e^{in\pi}}{2\pi(1-in)}\left[e^{\pi} - e^{-\pi}\right] \quad \frac{(-1)^n}{2\pi(1-in)}\left[e^{\pi} - e^{-\pi}\right]$$

Note: $\dfrac{1}{1-in} = \dfrac{1}{1-in}\,\dfrac{1+in}{1+in} = \dfrac{1+in}{1+n^2} \Rightarrow \boxed{c_n = \dfrac{(-1)^n\left(e^{\pi} - e^{-\pi}\right)}{2\pi\left(1+n^2\right)}(1+in)}$

Now, derive the <u>real</u> Fourier coefficients:

$$a_n = c_n + c_{-n} = \frac{(-1)^n(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}$$

$$b_n = i(c_n - c_{-n}) = \frac{-(-1)^n n(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}$$

<u>Parseval's identity</u>: If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$ [<u>Note</u>: this is just $\langle f(x), f(x)\rangle$!]

<u>Proof</u>: $$\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\Big(\underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx}_{f(x)}\Big)\,dx$$

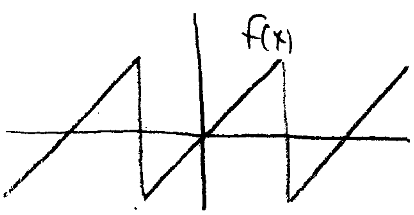
$$= \frac{a_0}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx + \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\,dx$$

$$= \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\Big(a_n \cdot \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}_{a_n} + b_n \cdot \underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx}_{b_n}$$

$$= \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

<u>Neat application</u>: Compute $\sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots$

Let $f(x) = x$ on $[-\pi, \pi]$. $a_n = 0$ (since $f(x)$ is odd)



$b_n = -\frac{2}{n}(-1)^n$ (Example 2, p. 5-6)

$\Rightarrow b_n^2 = \frac{4}{n^2}$

Apply Parseval's identity: $\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x))^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{2}{3}\pi^2$ (<u>LHS</u>)

<u>RHS</u>: $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty}\frac{4}{n^2}$

Equate LHS & RHS: $\sum_{n=1}^{\infty}\frac{4}{n^2} = \frac{2\pi^2}{3} \Longrightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$