MTHSC 208, Summer II, 2010

Week 2 summary:

- 3 ways to solve linear inhomogeneous ODE's: $y'(t) = a(t)\,y(t) + f(t)$

  (i) Integrating factor: $y' - ay = f$, int. factor $= e^{-\int a(t)\,dt}$

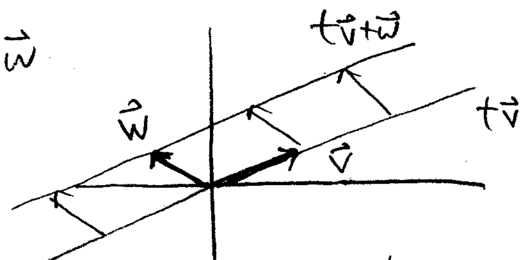
  "product rule in reverse"

  (ii) Variation of parameters: $y(t) = v(t)\,y_h(t)$; solve for $v(t)$.

  (iii) Homogeneous + particular sol'n: $y(t) = y_h(t) + y_p(t)$

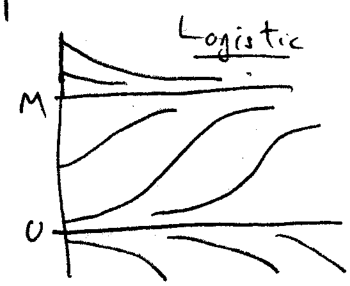
- Connection between general sol'n to linear ODE's & parametrized (linear) lines

$$y(t) = C\,y_h(t) + y_p(t), \qquad \vec{c}(t) = t\vec{v} + \vec{w}$$

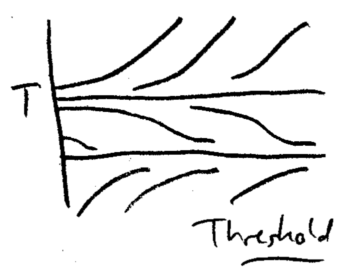


- Mixing problems: $x'(t) = (\text{rate in}) - (\text{rate out})$.

- Logistic equation: $y'(t) = r\left(1 - \frac{y}{M}\right)y$, $\quad y(t) = \frac{M}{1 + Ce^{-rt}}$

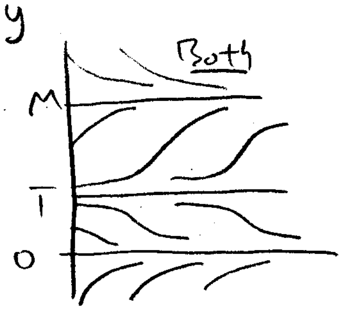


"Threshold equation": $y' = -r\left(1 - \frac{y}{T}\right)y$



Put these together:

$$y' = -r\left(1 - \frac{y}{M}\right)\left(1 - \frac{y}{T}\right)y$$



- 2nd order linear ODEs: $y'' + p(t)y' + q(t)y = f(t)$

  General sol'n: $y(t) = y_h(t) + y_p(t) = C_1 y_1(t) + C_2 y_2(t) + y_p(t)$

- 2nd order linear ODEs, constant coefficients: $y'' + py' + qy = f(t)$

* Homogeneous: $(f(t)=0)$ Assume sol'n has the form $y(t) = e^{rt}$.

  Plug back in & solve for r. Get $e^{rt}(r^2+pr+q)=0$.

  3 cases: (i) $r_1 \neq r_2$ real. $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

  (ii) $r_{1,2} = r$ $\quad y(t) = C_1 e^{rt} + C_2 t e^{rt}$

  (iii) $r_{1,2} = a \pm bi$ $\quad y(t) = e^{at}(C_1 \cos bt + C_2 \sin bt)$

* Inhomogeneous: $(f(t) \neq 0)$. Method of undetermined coefficients.

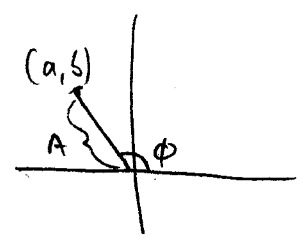
  Big idea: Guess $y_p(t)$ to have the same term as the forcing term, $f(t)$. Then, $y(t) = \bar{y}_h(t) + y_p(t)$.

$f(t)$	guess for $y_p(t)$
e^{rt}	ae^{rt}
$\cos wt$ or $\sin wt$	$a \cos wt + b \sin wt$
degree-n polynomial	degree-n polynomial
$e^{rt}\cos wt$ or $e^{rt}\sin wt$	$e^{rt}(a \cos wt + b \sin wt)$
linear combin. of above f'ns	linear combin. of above f'ns

- Simple harmonic motion: $x'' = -w^2 x$ has solution

  $x(t) = a \cos wt + b \sin wt = A \cos\left[w\left(t - \frac{\phi}{w}\right)\right]$

  where $A = \sqrt{a^2+b^2}$, $\phi = \arctan(b/a)$



- General harmonic motion (mechanical systems)

  $x'' + 2cx' + w_0^2 x = f(t)$

  damping coeff. — spring const. — driving force

* With <u>damping</u> $(c \neq 0)$. Roots of char. eq'n: $r_{1,2} = -c \pm \sqrt{c^2 - \omega_0^2}$

<u>Case 1</u>: $c > \omega_0$ overdamped $x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

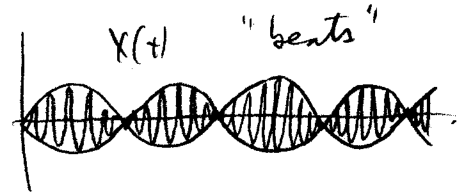
<u>Case 2</u>: $c < 0$ underdamped $x(t) = e^{-ct}(a \cos \omega t + b \sin \omega t)$, $\omega = \sqrt{c^2 - \omega_0^2}$

<u>Case 3</u>: $c = \omega_0$ critically damped $x(t) = C_1 e^{rt} + C_2 t e^{rt}$

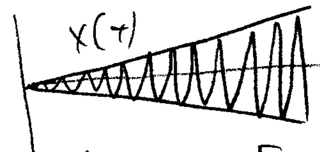
* <u>Forced harmonic motion:</u> $(f(t) \neq 0)$

e.g., $x'' + \omega_0^2 x = A \cos \omega t$, $x(0) = x'(0) = 0$

<u>Case 1</u>: $\omega \neq \omega_0$ $x(t) = \left(\frac{A}{2\delta \bar{\omega}} \sin \delta t\right) \sin \bar{\omega} t$



<u>Case 2</u> $\omega = \omega_0$ $x(t) = \frac{A}{2\omega_0} t \sin \omega_0 t$



"resonance frequency"

• <u>Basic linear algebra</u>

* A system of 2 linear equations $\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$

can be written as $A\vec{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \vec{b}$.

* $A\vec{x} = \vec{b}$ has a unique solution iff $\det A := a_{11}a_{22} - a_{12}a_{21} \neq 0$.

* The <u>inverse</u> of A exists iff $\det A \neq 0$, and is $A^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$,

and $AA^{-1} = A^{-1}A = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the identity matrix I.

* For any matrix A, $AI = IA = A$. Thus if $\det A \neq 0$, we can solve $A\vec{x} = \vec{b}$ by $\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{b}$.