Week 4 summary:

- Laplace transforms: $\mathcal{L}\{y(t)\}(s) = \int_0^\infty y(t)\, e^{-st}\, dt = Y(s)$

  Turns derivatives into multiplication by s.

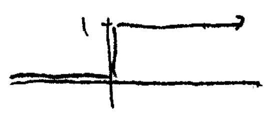
$$\begin{tikzcd} \text{ODE} \arrow[r, "\mathcal{L}"] \arrow[d, dashed] & \text{alg. eq'n} \arrow[d] \\ \text{ODE sol'n} & \text{sol'n} \arrow[l, "\mathcal{L}^{-1}"'] \end{tikzcd}$$

$$\mathcal{L}(y') = sY - y(0)$$

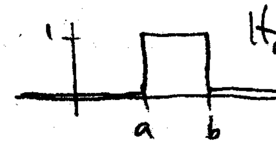
$$\mathcal{L}(y'') = s^2 Y - s\,y(0) - y'(0)$$

$$\vdots$$

- Piecewise continuous functions can be written concisely using Heavyside functions:



$H(t)$



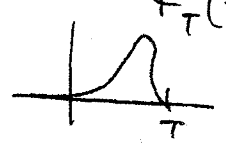
$H_{ab}(t) = H(t-a) - H(t-b)$

- $\mathcal{L}\{e^{ct} f(t)\}(s) = F(s-c)$

  $\mathcal{L}\{f(t-c)H(t-c)\}(s) = e^{-cs} F(s)$ if $c \geq 0$.

- If $f(t)$ is periodic with period T and "window"

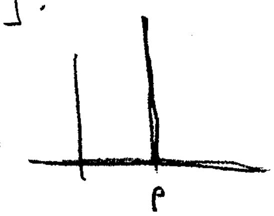
$F_T(t)$



,

  then $\mathcal{L}\{f(t)\}(s) = \dfrac{F_T(s)}{1-e^{-sT}} = F_T(s)\left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \dots\right].$

- Delta "function": $\delta_p(t) = \begin{cases} \text{"}\infty\text{"} & t = p \\ 0 & t \neq p \end{cases} = \lim_{\varepsilon \to 0} \delta_p^\varepsilon(t)$



  * $\int_{-\infty}^{\infty} \delta_0(t)\, dt = 1 \implies \int_{-\infty}^{\infty} \delta_p(t) f(t)\, dt = f(p)$

  * $\mathcal{L}(\delta_0(t)) = 1 \implies \mathcal{L}(\delta_p(t)) = e^{-sp}.$

- <u>Fourier series</u>: If $f(x)$ is $2L$-periodic, then we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$, $\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

- These formulas arose because we defined $\langle f(x), g(x)\rangle = \frac{1}{L}\int_{-L}^{L} f(x)\, g(x)\, dx$; we are "projecting" $f(x)$ onto the cosine & sine functions:

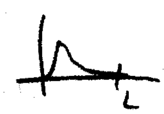
$$a_n = \langle f(x), \cos nx\rangle, \qquad b_n = \langle f(x), \sin nx\rangle.$$

- <u>Even functions</u>: $f(x) = f(-x)$
  - * Symmetric about y-axis.
  - * Fourier series contains only cosines
  - * $\int_{-L}^{L} f(x)\, dx = 2\int_0^L f(x)\, dx$

- <u>Odd functions</u>: $f(x) = -f(-x)$
  - * Symmetric about origin
  - * Fourier series contains only sines
  - * $\int_{-L}^{L} f(x)\, dx = 0$

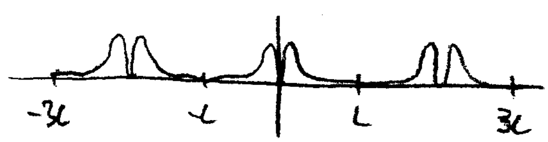
- <u>Fourier cosine & sine series</u>:

Start with a function $f(x)$ on $[0, L]$, e.g.,



* Fourier <u>cosine series</u> is the Fourier series of the <u>even</u> extension.

e.g.,

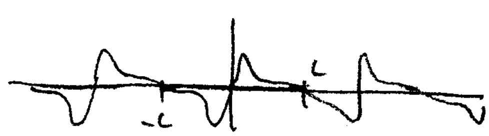


$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

* Fourier <u>sine series</u> is the Fourier series of the <u>odd</u> extension.

e.g.,



$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

- Complex Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{-\frac{in\pi x}{L}} = C_0 + \sum_{n=1}^{\infty} C_n e^{-\frac{in\pi x}{L}} + C_{-n} e^{\frac{in\pi x}{L}}$$

$$C_n = \frac{1}{2\pi} \int_{-L}^{L} f(x)\, e^{-\frac{in\pi x}{L}}\, dx$$

$$a_n = C_n + C_{-n} \qquad C_n = \frac{a_n - ib_n}{2}, \quad C_{-n} = \frac{a_n + ib_n}{2i}$$

$$b_n = i(C_n - C_{-n})$$

- Partial differential equations (PDE's): Equations involving a multivariate function & its partial derivatives.

  * Heat equation: $u_t = c^2 u_{xx}$, with 2 boundary conditions and 1 initial condition.

  * Solve using separation of variables: Assume $u(x,t) = f(x)g(t)$. and plug back in, solve for f & g.
    Finally, plug in $t=0$ & set equal to the initial condition.
    Use Fourier sine / cosine series.