We will solve for the function $u(x,y,t)$ defined for $0 \leq x, y \leq \pi$ and $t \geq 0$ which satisfies the following initial value problem of the heat equation:

$$u_t = c^2(u_{xx} + u_{yy})$$

$$u(0, y, t) = u(\pi, y, t) = u(x, 0, t) = u(x, \pi, t) = 0,$$

$$u(x, y, 0) = 2 \sin x \sin 2y + 3 \sin 4x \sin 5y.$$

(a) Carefully describe (and sketch) a physical situation that this models.

(b) Assume that $u(x, y, t) = f(x, y)g(t)$. Compute $u_{xx}$, $u_{yy}$, and $u_t$, find boundary conditions for $f(x, y)$. 
(c) Plug $u = fg$ back into the PDE and separate variables by dividing both sides of the equation by $c^2 fg$. Set this equal to a constant $\lambda$, and write down two equations: an ODE for $g(t)$, and a PDE $f(x, y)$ (called the *Helmholtz equation*), with four boundary conditions.

(d) Solve the ODE for $g(t)$.

(e) To solve the PDE for $f$, assume that $f(x, y) = X(x)Y(y)$. Plug this back in and separate variables. [For consistency, put the $X''/X$ term on one side of the equation, and set equal to a constant $\mu$.]
(f) Write down two ODEs – one for $X(x)$ and one for $Y(y)$, and include boundary conditions for both.  
*Hint:* It is easier notationally if you introduce a new constant, $\nu := \lambda - \mu$.

(g) Solve the ODEs for $X(x)$ and $Y(y)$, and determine $\mu$ and $\nu$ (and hence $\lambda$). You should get a $\lambda$ for each choice of positive integers $n, m \in \mathbb{N}$, call it $\lambda_{nm}$.

(h) For each $n, m \in \mathbb{N}$, we have a solution $u_{nm}(x, y, t) = f_{nm}(x, y)g_{nm}(t)$. Write down this solution.
(i) Find the general solution of the PDE. It will be a doubly infinite sum (superposition) of solutions:

\[ \sum_{n,m \in \mathbb{N}} u_{nm}(x, y, t). \]

(g) Find the particular solution to the initial value problem by using the initial condition.

(h) What is the long-term behavior of the system? Give a mathematical, and physical, justification.