1. Let $f, g \in \mathcal{B}[a,b]$ and $c \in \mathbb{R}$.

   (a) Prove that $\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b} (f+g)$, and find an example that satisfies

   $$\underline{\int_a^b} f + \underline{\int_a^b} g < \underline{\int_a^b} (f+g) = \overline{\int_a^b} (f+g) < \overline{\int_a^b} f + \overline{\int_a^b} g\,.$$

   (b) Prove $\overline{\int_a^b} (cf) = \begin{cases} c\overline{\int_a^b} f & \text{if } c \geq 0, \\ c\underline{\int_a^b} f & \text{if } c < 0, \end{cases} \qquad \underline{\int_a^b} (cf) = \begin{cases} c\underline{\int_a^b} f & \text{if } c \geq 0, \\ c\overline{\int_a^b} f & \text{if } c < 0, \end{cases}$

   (c) Prove that if $f \geq g$, then $\underline{\int_a^b} f \geq \underline{\int_a^b} g$ and $\overline{\int_a^b} f \geq \overline{\int_a^b} g$.

   (d) Prove $\left|\overline{\int_a^b} f\right| \leq \overline{\int_a^b} |f|$ and $\left|\underline{\int_a^b} f\right| \leq \overline{\int_a^b} |f|$. Find an example s.t. $\left|\underline{\int_a^b} f\right| \leq \underline{\int_a^b} |f|$.

2. Prove the following for $f \geq 0$:

   (a) If $f \in \mathcal{C}[0,1]$, then $\int_0^1 f = 0$ implies $f = 0$.

   (b) If we remove the continuity condition, then (a) need not hold.

   (c) If $f \in \mathcal{R}[0,1]$, then $f(x) = 0$ for all $x \in \mathbb{Q} \cap [0,1]$ implies $\int_0^1 f = 0$.

3. Let $0 < f \in \mathcal{R}[0,1]$. Prove or disprove the following:

   (a) $\displaystyle\int_0^1 f > 0\,,$ (b) $\displaystyle\frac{1}{f} \in \mathcal{R}[0,1]\,.$

4. Let $f \in \mathcal{C}[a,b]$. Prove that if $\int_a^b fg = 0$ for all $g \in \mathcal{R}[a,b]$, then $f = 0$.

5. Find a function $f \in \mathcal{B}[0,1]$ for which $f^2 \in \mathcal{R}[0,1]$ but $f \notin \mathcal{R}[0,1]$.

6. Using Lebesgue's theorem, show that $f + g \in \mathcal{R}[a,b]$ for any $f, g \in \mathcal{R}[a,b]$.