1. Let \( f(x) = \begin{cases} 
1, & x = 0 \\
\frac{1}{n}, & x = \frac{m}{n} \in \mathbb{Q} \cap (0, 1] \\
0, & x \in \mathbb{Q}^c \cap [0, 1]\end{cases} \) Prove that \( f \) is continuous at all irrational points.

2. Prove, for \( f \in \mathcal{B}[a, b] \) and a partition \( Q = \{y_0, y_1, \ldots, y_K\} \),
\[
\mathcal{U}(\mathcal{P}, f) - (K - 1)(M - m)||\mathcal{P}|| \leq \mathcal{U}(\mathcal{P} \cup Q, f), \quad \forall \mathcal{P}: \text{any partition}.
\]

Hint: First, draw an appropriate picture that contains the essential idea of the proof.

3. For \( f \in \mathcal{B}[a, b] \), prove that \( \int_{a}^{b} f = \lim_{||\mathcal{P}|| \to 0} \mathcal{U}(\mathcal{P}, f) \), where \( ||\mathcal{P}|| = \max \Delta x_i \).

4. Use the fundamental theorem of calculus to evaluate \( \int_{0}^{1} x \ln x \, dx \).

5. Let \( f(t) = \begin{cases} 
t, & t \in [0, 1) \\
-\frac{b}{t}, & t \in [1, 2].\end{cases} \)

(a) Find the indefinite integral, \( G(x) := \int_{0}^{x} f(t) \, dt \) for all \( x \in [0, 2] \).

(b) For what value of \( b \) is \( G(x) \) differentiable for all \( x \in [0, 2] \)?

6. Find \( F'(x) \), where \( F \) is defined on \( [0, 1] \) as follows:

(a) \( F(x) = \int_{x}^{1} \sqrt{1 + t^3} \, dt \).

(b) \( F(x) = \int_{0}^{x^2} f(t) \, dt \), where \( f \) is continuous.

(c) \( F(x) = \int_{h(x)}^{g(x)} f(t) \, dt \), where \( f \) is continuous and \( g \) and \( h \) are differentiable.

(d) \( F(x) = \int_{x-a}^{x+a} f(t) \, dt \), where \( f \) is continuous and \( a > 0 \).

7. Let \( f \in \mathcal{C}[a, b] \) and \( g \in \mathcal{R}[a, b] \) with \( g \geq 0 \). Prove that
\[
\exists c \in [a, b] \text{ such that } \int_{a}^{b} f(x)g(x) \, dx = f(c) \int_{a}^{b} g(x) \, dx.
\]

8. Let \( f \in \mathcal{C}[0, 1] \). Prove that \( \lim_{n \to \infty} \int_{0}^{1} f(x^n) \, dx = f(0) \).