1. Find the following integrals. They may or may not exist depending on $p \in \mathbb{R}$.

$$\text{(a)} \int_0^1 x^p \, dx\,, \qquad \text{(b)} \int_1^\infty x^p \, dx\,, \qquad \text{(c)} \int_0^\infty x^p \, dx\,.$$

2. The Gamma function is defined by $\Gamma(x) = \displaystyle\int_0^\infty e^{-t} t^{x-1} \, dt$ for $x \in (0, \infty)$. Show that $\Gamma(n+1) = n!$ for all $n \in \mathbb{N}$.

3. Let $\alpha(x) = \begin{cases} -1, & x \in [-1, 0), \\ 0, & x = 0 \\ 1, & x \in (0, 1] \end{cases}$ and $f \in \mathcal{B}[-1, 1]$ such that f is continuous at 0. Evaluate $\displaystyle\int_{-1}^1 f \, d\alpha$.

4. Let $\alpha(x) = \displaystyle\sum_{n=1}^\infty \frac{1}{2^n} H\left(x - \frac{1}{n}\right)$. Evaluate the following integrals.

$$\text{(a)} \int_0^1 f \, d\alpha \ \text{ for } f \in \mathcal{C}[0, 1]\,, \qquad \text{(b)} \int_0^1 x \, d\alpha\,, \qquad \text{(c)} \int_0^1 \alpha(x) \, dx\,.$$

5. Let $\alpha(x) = H(x) + H(x - 1)$ on $[-1, 1]$ where $H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0 \end{cases}$

   (a) Find $\displaystyle\int_{-1}^1 x^2 \, d\alpha$ without using integration by parts.

   (b) Verify your answer in (a) using integration by parts.

   (c) Determine whether $\displaystyle\int_{-1}^1 \alpha(x) \, d\alpha(x)$ exists or not.

   (d) For (c), we might try to use integration by parts as follows:

$$\int_{-1}^1 \alpha \, d\alpha = \alpha^2 \Big|_{-1}^1 - \int_{-1}^1 \alpha \, d\alpha \qquad \Longrightarrow \qquad \int_{-1}^1 \alpha \, d\alpha = \frac{2^2 - 0^2}{2} = 2\,,$$

   which is false. What is wrong in the above argument?

6. Find the following integrals if they exist, where $\widetilde{H}(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$

$$\text{(a)} \int_{-1}^1 H \, d\widetilde{H}\,, \qquad \text{(b)} \int_{-1}^1 \widetilde{H} \, dH\,.$$

7. Find the following integrals if they exist:

$$\text{(a)} \int_0^3 [x] \, dx^2\,, \qquad \text{(b)} \int_0^3 x^2 \, d[x]\,, \qquad \text{(c)} \int_1^3 ([x] + x) \, d(x^2 + e^x)\,.$$