1. Complete the following proofs that were skipped in lecture:

   (a) In the proof of the ratio test, prove that $r > 1 \implies \sum_{n=1}^{\infty} a_n = \infty$.

   (b) In the proof of the root test, prove that $\alpha > 1 \implies \sum_{n=1}^{\infty} a_n = \infty$.

   (c) Prove that $r \leq \underline{\lim}_{n\to\infty}(a_n)^{1/n}$.

2. Determine convergence or divergence of the following infinite series:

   (a) $\sum_{n=1}^{\infty} n^3 e^{-n}$ (b) $\sum_{n=1}^{\infty} \frac{3^n}{n!}$ (c) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ (d) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$

   (e) $\sum_{n=1}^{\infty} \frac{p(n)}{a^n}$, $p(x)$ polynomial, $a > 1$ (f) $\sum_{n=2}^{\infty} \frac{\ln n}{n}$ (g) $\sum_{n=1}^{\infty} \cos\left(\frac{1}{n^p}\right)$ $p > 0$.

3. Determine $p, q \in \mathbb{R}$ for which the following infinite series converges:

   (a) $\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$, $a, b > 0$ (b) $\sum_{n=1}^{\infty} (\sin p)^n$

4. Apply the root and ratio tests to the series $\sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} 2^{-n}, & n \text{ is even} \\ 2^{-(n+2)}, & n \text{ is odd.} \end{cases}$

5. Give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} (a_{2n-1} + a_{2n}) \quad \text{converges,} \qquad \text{but} \quad \sum_{n=1}^{\infty} a_n \quad \text{diverges.}$$

6. Let $a_n \geq 0$ and $\sum_{n=1}^{\infty} a_n < \infty$. For each of the following, either prove that the given series converges, or give an example for which the series diverges.

   (a) $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$, (b) $\sum_{n=1}^{\infty} \sqrt{a_n}$ (c) $\sum_{n=1}^{\infty} \sqrt{\frac{a_n}{n}}$.

7. Prove the following trigonometric identities which were skipped in lecture.

   (a) $\sum_{k=1}^{n} \sin(kt) \frac{\cos\frac{t}{2} - \cos(n+\frac{1}{2})t}{2\sin\frac{t}{2}}, \quad \forall t \in \mathbb{R} \setminus \{2\pi m\}_{m\in\mathbb{Z}}$.

   (b) $\sum_{k=1}^{n} \cos(kt) \frac{\sin(n+\frac{1}{2})t - \sin\frac{t}{2}}{2\sin\frac{t}{2}}, \quad \forall t \in \mathbb{R} \setminus \{2\pi m\}_{m\in\mathbb{Z}}$.

8. Prove or disprove (Compare these with Abel's test):

   (a) Let $b_k \to 0$. Then $\displaystyle\sum_{k=1}^{\infty} a_k < \infty \implies \sum_{k=1}^{\infty} a_k b_k < \infty$.

   (b) Let $b_k \to b \neq 0$. Then $\displaystyle\sum_{k=1}^{\infty} a_k < \infty \implies \sum_{k=1}^{\infty} a_k b_k < \infty$.