1. Prove the conditional convergence of each of the following series:

$$\text{(a)} \quad \sum_{n=1}^{\infty}(-1)^n\frac{\ln n}{n} \qquad\qquad \text{(b)} \quad \sum_{n=1}^{\infty}\frac{\sin n}{n}$$

2. Suppose $\lim\limits_{n\to\infty} na_n = A \neq 0$. Prove that $\sum\limits_{n=1}^{\infty} a_n$ diverges.

3. Determine the convergence/divergence of the following series:

   (a) $1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}+\frac{1}{8}-\frac{1}{9}+\cdots$.

   (b) $1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$.

4. If $\sum\limits_{n=1}^{\infty} a_n$ converges conditionally, prove there exist rearrangements a_n' and $\widetilde{a}_n$ such that

$$\sum_{n=1}^{\infty} a_n' = \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} \widetilde{a}_n = -\infty\,.$$

5. Prove the following:

   (a) $\sum\limits_{n=1}^{\infty}|a_n|^2<\infty$ and $\sum\limits_{n=1}^{\infty}|b_n|^2<\infty \implies \sum\limits_{n=1}^{\infty}a_nb_n$ converges absolutely.

   (b) $\sum\limits_{n=1}^{\infty}|a_n|<\infty$ and $\sum\limits_{n=1}^{\infty}b_n<\infty \implies \sum\limits_{n=1}^{\infty}a_nb_n$ converges absolutely.

6. Prove that $\ell^1 \subsetneq \ell^2 \subsetneq \ell^\infty$.

7. Prove that the following are normed linear spaces:

   (a) $(\ell^p, ||\cdot||_p)$ for $p=1,2,\infty$.

   (b) $(\mathcal{B}[0,1], ||\cdot||_\infty)$, $(\mathcal{R}[0,1], ||\cdot||_\infty)$, $(\mathcal{C}[0,1], ||\cdot||_\infty)$.

8. Find the pointwise limit of each of the following sequences of functions:

$$\text{(a)} \quad f_n(x)=\frac{\sin nx}{1+nx} \text{ on } [0,\infty) \qquad\qquad \text{(b)} \quad f_n(x)=nxe^{-nx^2} \text{ on } \mathbb{R}.$$

9. Let $f_n\colon \mathbb{N}\to\mathbb{R}$ defined by $f_n(m)=\frac{n}{m+n}$. Show that $\lim\limits_{m\to\infty}\lim\limits_{n\to\infty} f_n(m) \neq \lim\limits_{n\to\infty}\lim\limits_{m\to\infty} f_n(m)\,.$

10. For $n\geq 2$, define $f_n(x)=\begin{cases} n^2x & \text{on } [0,\frac{1}{n}], \\ 2n-n^2x & \text{on } (\frac{1}{n},\frac{2}{n}], \\ 0 & \text{on } (\frac{2}{n},1], \end{cases}$ Sketch $f_n(x)$ and show that

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx\,.$$