MthSc 208: Differential Equations (Summer II, 2012)
In-class Worksheet 7c: The 2D Heat Equation

NAME:

We will solve for the function $u(x,y,t)$ defined for $0 \leq x, y \leq \pi$ and $t \geq 0$ which satisfies the following initial value problem of the heat equation:

$$u_t = c^2(u_{xx} + u_{yy}) \qquad \begin{aligned} &u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0, \\ &u(x,y,0) = 2\sin x \sin 2y + 3\sin 4x \sin 5y\,. \end{aligned}$$

(a) Carefully descsribe (and sketch) a physical situation that this models.

(b) Assume that $u(x,y,t) = f(x,y)g(t)$. Compute u_{xx}, u_{yy}, and u_t, find boundary conditions for $f(x,y)$.

(c) Plug $u = fg$ back into the PDE and separate variables by dividing both sides of the equation by $c^2 fg$. Set this equal to a constant λ, and write down two equations: an ODE for $g(t)$, and a PDE $f(x, y)$ (called the *Helmholtz equation*), with four boundary conditions.

(d) Solve the ODE for $g(t)$.

(e) To solve the PDE for f, assume that $f(x, y) = X(x)Y(y)$. Plug this back in and separate variables. [For consistency, put the X''/X term on one side of the equation, and set equal to a constant μ.]

(f) Write down two ODEs – one for $X(x)$ and one for $Y(y)$, and include boundary conditions for both. *Hint*: It is easier notationally if you introduce a new constant, $\nu := \lambda - \mu$.

(g) Solve the ODEs for $X(x)$ and $Y(y)$, and determine μ and ν (and hence λ). You should get a λ for each choice of positive integers $n, m \in \mathbb{N}$, call it λ_{nm}.

(h) For each $n, m \in \mathbb{N}$, we have a solution $u_{nm}(x, y, t) = f_{nm}(x, y)g_{nm}(t)$. Write down this solution.

(i) Find the general solution of the PDE. It will be a doubly infinite sum (superposition) of solutions: $\displaystyle\sum_{n,m\in\mathbb{N}} u_{nm}(x,y,t)$.

(g) Find the particular solution to the initial value problem by using the initial condition.

(h) What is the long-term behavior of the system? Give a mathematical, and physical, justification.