Read: Rudin, Chapter 1. Do not wory about understanding every detail, but get a good sense of the big ideas.

1. In no more than four sentences, summarize the main concepts and ideas of Chapter 1 in Rudin. Pretend that you are writing for a friend who is an English major (and naturally, took math throughout high school, but maybe not in college). They should be able to understand it *and* appreciate the clarity and style in which it is written.

2. Suppose that we were to define a cut *without* the requirement that it contain no largest element. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, the addition satisfies axioms (A1) to (A4) [see Definition 1.12 in Rudin]. What is the zero-element? Also, show that (A5) fails.

3. Fix $b > 1$. This problem builds off of Theorem 1.21 in Rudin, which defined (and proved the existence of) for each $n \in \mathbb{N}$, the (positive) n^{th} root of b, as

$$b^{1/n} := \sup\{x \in \mathbb{Q} : x^n < b\}\,.$$

You may use the result of the Corollary to this theorem (see page 11 of Rudin): If a and b are positive real numbers and $n \in \mathbb{N}$, then $(ab)^{1/n} = a^{1/n}b^{1/n}$.

(a) If $m, n, p, q \in \mathbb{Z}$, and $n, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}\,.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$, i.e., this definition is *well-defined.*

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x.

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.

4. Prove that no order can be defined in the complex field that turns it into an ordered field. [*Hint*: -1 is a square.]

5. Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?

6. If $z_1, \ldots, z_n$ are complex, prove that $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$.

7. Under what conditions does equality hold in the Cauchy-Schwarz inequality? Justify your answer.