Read: Rudin, Chapter 2, pages 36–40.

Throughout, let X be a metric space. Recall that a set $E \subset X$ is *dense* if every point of X is either in E or is a limit point of E. Equivalently, E is dense in X if $\bar{E} = X$. A metric space is called *separable* if it contains a countable dense subset.

A collection $\{V_\alpha\}$ of open subsets of X is said to be a *basis* for X if the following is true:

> For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α. (In other words, every open set in X is the union of a subcolection of $\{V_\alpha\}$.)

1. Show that $\mathbb{R}^k$ is separable. [*Hint*: Consider points which only have rational coordinates – prove this set is dense.]

2. Let X be a separable metric space.
   (a) Prove that X has a *countable* basis. [*Hint*: Take all neighborhoods with rational radius and center in some countable dense subset of X.]
   (b) Prove that every open cover of X has a *countable* subcover.
   (c) Deduce that every open set in $\mathbb{R}$ is the union of an at most countable collection of disjoint open segments.

3. Let X be a metric space in which every infinite subset has a limit point.
   (a) Show that for any $\delta > 0$, X can be covered by finitely many neighborhoods of radius δ. [*Hint*: Pick the centers inductively: Start with any $x_1 \in X$. Having chosen $x_1, \dots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$. Show that this process must stop after a finite number of steps.]
   (b) Prove that X is separable. [*Hint*: Take $\delta_n = 1/n$ for $n = 1, 2, 3 \dots$ and consider the centers of the corresponding neighborhoods. Show that this set is dense in X.]

4. Theorem 2.37 in Rudin says that every infinite subset of a compact set has a limit point. Prove the converse: If X is a metric space in which every infinite subset has a limit point, then X is compact. [*Hint*: By Problems 2 and 3, X is separable, and hence every open cover of X has a *countable* subcover $\{G_n\}$, $n = 1, 2, 3, \dots$. If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $G_1 \cup \dots \cup G_n$ is nonempty for each n, but $\cap F_n$ is empty. If E is a set which contains a point from each F_n, consider a limit point of E, and obtain a contradiction.]

5. Let X be a compact metric space. Prove that X is separable, and deduce that X has a countable basis.