*Read*: Rudin, Chapter 2, pages 41–43.

1. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space with $d(p,q) = |p-q|$. Let $E$ be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that $E$ is closed and bounded in $\mathbb{Q}$, but that $E$ is not compact. Is $E$ open in $\mathbb{Q}$? [*Hint*: Use Theorems 2.30 and 2.33 in Rudin.]

2. Let $E$ be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is $E$ countable? Is $E$ dense in $[0,1]$? Is $E$ compact? Is $E$ perfect? Prove all of your claims. [A set $E$ is *perfect* if it is closed, and every point is a limit point of $E$.]

3. Consider the set $S = \mathbb{Q} \cap [0,1]$, which we know is countable, Enumerate $S$ via a function $f \colon \mathbb{N} \to \mathbb{Q}$, so that $S = \{f(1), f(2), \dots\}$. Define the set $P \subset S$

$$P = \{0.d_1 d_2 d_3 \ldots \in S : d_i \neq f(n)_n\},$$

where $0.d_1 d_2 d_3 \ldots$ is the base-10 decimal representation of a number (assuming it does not end in an infinite string of 9s), and $f(n)_n$ is the $n^{\text{th}}$ digit of the decimal representation of $f(n)$. By construction, $P \cap \mathbb{Q} = \emptyset$. Prove that $P$ is perfect, and deduce that there are nonempty perfect sets in $\mathbb{R}$ that contain no rational number?

4. Let $X$ be a metric space.
   (a) If $A$ and $B$ are disjoint closed sets in $X$, prove that they are separated.
   (b) Prove the same for disjoint open sets.
   (c) Fix $p \in X$, $\delta > 0$, and define $A$ to be the set of all $q \in X$ for which $d(p,q) < \delta$. Define $B$ similarly, but with $>$ in place of $<$. Prove that $A$ and $B$ are separated.
   (d) Prove that every connected metric space with at least two points is uncountable. [*Hint*: Use (c).]

5. Are closures and interiors of connected sets always connected? Prove or disprove each assertation.