*Read*: Rudin, Chapter 3, pages 47–65.

1. For this problem, consider the metric space $X = \mathbb{C}$.
   (a) Show that $\big||z| - |w|\big| \leq |z - w|$ for all $z, w \in \mathbb{C}$. [*Hint*: By the triangle inequality, $|z| = |z - w + w| \leq |z - w| + |w|$.]
   (b) Prove that convergence of $\{z_n\}$ implies convergence of $\{|z_n|\}$. Show by example that the converse need not hold.

2. Put $p_1 = \sqrt{2}$ and recursively define a sequence $\{p_n\}$ by
$$p_{n+1} = \sqrt{2 + \sqrt{p_n}} \qquad (n = 1, 2, 3, \dots)\,.$$
Prove that $\{p_n\}$ is monotonically increasing and bounded above by 2, from which we can deduce that it converges.

3. Consider the sequence $\{a_n\}$ defined by
$$a_1 = 0\,, \qquad a_{2m} = \frac{a_{2m-1}}{2}\,, \qquad a_{2m+1} = \frac{1}{2} + a_{2m}\,.$$
   (a) Write out the first 10 terms of this sequence. Make a conjecture for what $a_{2n}$ and $a_{2n+1}$ are for all $n$.
   (b) Prove your conjectures by induction.
   (c) Find all subsequential limits of $\{a_n\}$, and determine $\limsup a_n$ and $\liminf a_n$.

4. For any two sequences $\{a_n\}$ and $\{b_n\}$ of real numbers, prove that
$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n\,,$$
provided that the sum on the right is not of the form $\infty - \infty$. Give an explicit example of where equality does not hold.

5. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges. [*Hint*: Define $c_n = |b_n - b|$, where $b_n \to b$ (how do you know that $b$ exists?) and use the comparison test.] Additionally, give examples to show how this can fail if *either* the "monotonic" or "bounded" condition is dropped from the hypothesis.

6. Let $\{p_n\}$ be a sequence of real numbers, and define its arithmetic means $\sigma_n$ by
$$\sigma_n = \frac{p_0 + p_1 + \cdots + p_n}{n+1} \qquad (n = 0, 1, 2, \dots)\,.$$
   (a) If $\lim p_n = p$, prove that $\lim \sigma_n = p$.
   (b) Construct a sequence $\{p_n\}$ which does not converge, although $\lim \sigma_n = 0$.
   (c) Can it happen that $p_n > 0$ for all $n$ and that $\limsup p_n = \infty$, although $\lim \sigma_n = 0$?