Read: Rudin, Chapter 3, pages 47–65.

1. For this problem, consider the metric space \( X = \mathbb{C} \).
   (a) Show that \( |z - w| \leq |z - w| \) for all \( z, w \in \mathbb{C} \). \([\text{Hint: By the triangle inequality,} \]
   \( |z| = |z - w + w| \leq |z - w| + |w| \).\]
   (b) Prove that convergence of \( \{z_n\} \) implies convergence of \( \{|z_n|\} \). Show by example that
   the converse need not hold.

2. Put \( p_1 = \sqrt{2} \) and recursively define a sequence \( \{p_n\} \) by
   \[ p_{n+1} = \sqrt{2 + \sqrt{p_n}} \quad (n = 1, 2, 3, \ldots) \, . \]
   Prove that \( \{p_n\} \) is monotonically increasing and bounded above by 2, from which we can
deduce that it converges.

3. Consider the sequence \( \{a_n\} \) defined by
   \[ a_1 = 0, \quad a_{2m} = \frac{a_{2m-1}}{2}, \quad a_{2m+1} = \frac{1}{2} + a_{2m} \, . \]
   (a) Write out the first 10 terms of this sequence. Make a conjecture for what
   \( a_{2n} \) and \( a_{2n+1} \) are for all \( n \).
   (b) Prove your conjectures by induction.
   (c) Find all subsequential limits of \( \{a_n\} \), and determine \( \limsup a_n \) and \( \liminf a_n \).

4. For any two sequences \( \{a_n\} \) and \( \{b_n\} \) of real numbers, prove that
   \[ \limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n \, , \]
   provided that the sum on the right is not of the form \( \infty - \infty \). Give an explicit example
   of where equality does not hold.

5. If \( \sum a_n \) converges, and if \( \{b_n\} \) is monotonic and bounded, prove that \( \sum a_n b_n \) converges.
   \([\text{Hint: Define} \ c_n = |b_n - b|, \ \text{where} \ b_n \to b \ \text{how do you know that} \ b \ \text{exists?} \] and use
   the comparison test.] Additionally, give examples to show how this can fail if \( \text{either} \) the
   “monotonic” or “bounded” condition is dropped from the hypothesis.

6. Let \( \{p_n\} \) be a sequence of real numbers, and define its arithmetic means \( \sigma_n \) by
   \[ \sigma_n = \frac{p_0 + p_1 + \cdots + p_n}{n + 1} \quad (n = 0, 1, 2, \ldots) \, . \]
   (a) If \( \lim p_n = p \), prove that \( \lim \sigma_n = p \).
   (b) Construct a sequence \( \{p_n\} \) which does not converge, although \( \lim \sigma_n = 0 \).
   (c) Can it happen that \( p_n > 0 \) for all \( n \) and that \( \limsup p_n = \infty \), although \( \lim \sigma_n = 0 \)?