Read: Rudin, Chapter 3, pages 65–78 and exercises 24, 25.

1. Let X be a complete metric space. For a nonempty set $E \subset X$, define the *diameter* of E by
$$\operatorname{diam} E = \sup\{d(p,q) : p,q \in E\}\,.$$
   (a) Let $\{E_n\}$ be a sequence of nested closed nonempty bounded sets. Prove that if $\lim\limits_{n\to\infty} \operatorname{diam} E_n = 0$, then $\cap_{n=1}^{\infty} E_n$ consists of exactly one point. [*Hint*: First show that it is nonempty, then show that if it contains $p, q \in X$, then $p = q$.]
   (b) Prove the *Baire category theorem*: If $\{G_n\}$ is a sequence of dense open subsets of X, then $\cap_{n=1}^{\infty} G_n$ is nonempty. (In fact, it is dense in X, but you do not need to prove this.) [*Hint*: Find a nested sequence of neighborhoods E_n such that $\bar{E}_n \subset G_n$ and appeal to Part (a).]

2. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges. [*Hint*: You are trying to show that for any $\epsilon > 0$, there is some N such that for all $n, m \geq N$,
$$|d(p_n, q_n) - d(p_m, q_m)| < \epsilon\,.$$
Notice that by the triangle inequality,
$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$$
holds for all n and m.]

3. Let X be a metric space.
   (a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ *equivalent* if
$$\lim_{n\to\infty} d(p_n, q_n) = 0\,.$$
   Prove that this is an equivalence relation.
   (b) Let X^* be the set of all such equivalence classes. If $P, Q \in X^*$, where $\{p_n\} \in P$ and $\{q_n\} \in Q$, define a distance function by
$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n)\,,$$
   which exists by the previous exercise. Show that this distance function is *well-defined*. That is, it does *not* depend on the choice of representative from P or Q.
   (c) We call X^* the *completion* of X. In a few sentences, summarize this construction and its significance. Your explanation should be understandable to a student who has taken taken MthSc 453 and thus knows what a Cauchy sequence and a complete metric space is, but did not learn about completions.

4. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{h \to 0} [f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}$. Does this imply that f is continuous?

5. If $f\colon X \to Y$ is a continuous mapping between metric spaces, prove that

$$f(\bar{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. Show by an example that $f(\bar{E})$ can be a proper subset of $\overline{f(E)}$.

6. Let $f\colon X \to \mathbb{R}$ be a continuous real-valued function on a metric space X. Let $Z(f)$ (the *zero set* of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.