MthSc 453

Lecture 1: Construction of the rational numbers.

"God created the integers, all else is the work of man." Kronecker 1886

Kronecker:
- Finite mathematics (Finite #'s & finite # of steps).
- Non-constructive proofs are skeptical
- Lindemann's proof that $\pi \notin \mathbb{Q}$: Nice, but proved nothing.

Weird things happen involving ∞. Ex 1: $1-1+1-1+1-1+/\ldots$

Example: Consider $\sum_{n=1}^{\infty} \frac{1}{n}(-1)^n = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+/-\cdots$

$$= 1-\underbrace{\left(\tfrac{1}{2}-\tfrac{1}{3}\right)}_{>0}-\underbrace{\left(\tfrac{1}{4}-\tfrac{1}{5}\right)}_{>0}-\underbrace{\left(\tfrac{1}{6}-\tfrac{1}{7}\right)}_{>0}-\left(\tfrac{1}{8}-\tfrac{1}{9}\right)-\cdots$$

$$< 1-\frac{1}{2}+\frac{1}{3} = \frac{5}{6}$$

But: Consider a rearrangement: $1+\frac{1}{3}-\frac{1}{2}+\underbrace{\frac{1}{5}+\frac{1}{7}-\frac{1}{4}}_{>0}+/-\underbrace{\ldots\ldots}_{>0} > \frac{5}{6}.$

How is this possible?

Morals:
- We need a better understanding of the fundamentals of mathematics.
- Some things which are "obvious" haven't always been, and vice-versa.

Newton (1666): Infinite series $= \pi$.

But what does this mean? Forms the basis of calculus.

What does it mean for a set of #'s to converge.

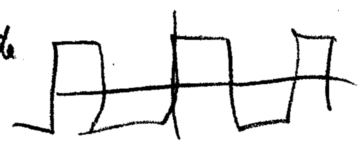
What is a limit?

Could a sequence of rational numbers converge but not have a limit?

Can we define what it means for a sequence to converge w/o the notion of a limit?

~~Early~~ Late 1600's: Newton, Leibniz developed calculus. The idea of a limit was there, but not made explicit.

Early 1800's: Fourier series invented. Laplace, Lagrange felt uneasy.

For example, we can write  as $\sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi} \sin nx$.

Is this cheating? But it gives answers! Seemed right.

Concepts became precise

Cauchy 1820's

Weierstraß / Riemann 1850's, 60's.

- We'll construct the rational numbers $\mathbb{Q}$ from $\mathbb{Z}$.

Sets & relations

~~Def~~: A set is a collection of objects. (no repeats allowed).

write $S = \{1,$  $, \{2, a\}\}$.

or $S = \{x : P(x) \text{ is true}\}$.

e.g., $\{x : x > 0\}$.

Notation: $x \in S$ — x is in S.

$x \notin S$ — x is not in S.

Notation (cont.):

$\emptyset$ is the empty set.

$A \subset B$ means "A is a subset of B," i.e., if $x \in A$, then $x \in B$.

or $x \in A \Rightarrow x \in B$.

Remark: $\subset$ and $\subseteq$ mean the same thing! (Not like $<$, $\leq$).

If $A \subset B$ and $B \not\subset B$, then A is a proper subset of B.

If $A \subset B$ and $B \subset A$, then $A = B$, else $A \neq B$.

More: $A \cup B = \{x : x \in A \text{ or } x \in B\}$. (union)

$A \cap B = \{x : x \in A \text{ and } x \in B\}$ (intersection)

$A^c = \{x : x \notin A\}$. (complement).

$A \setminus B = \{x : x \in A \text{ and } x \notin B\}$. (minus).

$A \times B = \{(a,b) : a \in A, \text{ and } b \in B\}$. (product)

$\uparrow$ ordered pair.

Relations:

Def: A (binary) relation R is a subset of $A \times B$.

If $(a,b) \in R$, write aRb.

Example: A "is an ancestor of" is a relation on $P \times P$ (people.)

L "likes" is a relation on $P \times P$.

S "is a sibling of" " " " "

$<$ "less than" " " $\mathbb{Z} \times \mathbb{Z}$ (integers.)

Def: An <u>equivalence relation</u> R on S is a relation on $S \times S$ s.t.

reflexive (1) aRa

symmetric (2) $aRb \Rightarrow bRa$

transitive (3) aRb and $bRc \Rightarrow aRc$.

We usually write these as e.g., $\approx$, $\sim$, $\cong$, $\simeq$, etc. (note $=$ is reserved.)

Remark: Any function $F: A \to B$ is a relation s.t.

if aFb and aFb' then $b = b'$. **Example** Birthdays!

We usually write $F(a) = b$; different notation.

Construction of $\mathbb{Q}$ (rational numbers)

Assume $\mathbb{Z}$, the integers, their arithmetic, and order

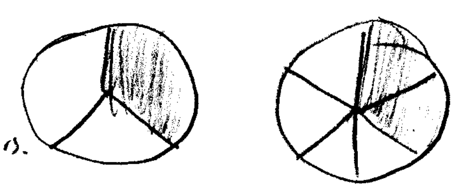
Question: what is $\mathbb{Q}$?

Fractions? Repeating decimals? What about $0.9999\ldots$

$\to \left\{ \frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0 \right\}$

Not a good answer. Problem: What does $\frac{m}{n}$ mean???

Motivation: $\frac{1}{3}$ is the same (equivalent) as $\frac{2}{6}$



Write: $(1,3) \sim (2,6)$ equivalent ordered pairs.

Idea: These belong to some <u>equivalence class</u>, call it "$\frac{1}{3}$"

(What's the equivalence relation???).

Let $\mathbb{Q}$ = set of all such equivalence classes of ordered pairs of $\mathbb{Z}\times\mathbb{Z}\setminus\{0\}$

Goal: Have these pairs extend $\mathbb{Z}$ so that $\frac{n}{1}\in\mathbb{Q} \leftrightarrow n\in\mathbb{Z}$.

Note that $\mathbb{Q} = \left\{ \frac{m}{n} : m,n\in\mathbb{Z},\ n\neq 0 \right\}$, where $\frac{m}{n}$ is an equivalent class of (m,n) w/ relation $(p,q)\sim(m,n)$ iff $\begin{cases} pn = qm \\ q,n\neq 0. \end{cases}$

Check: $\sim$ is an equivalence relation: (on $(\mathbb{Z}\times\mathbb{Z}\setminus\{0\})\times(\mathbb{Z}\times\mathbb{Z}\setminus\{0\})$)

HW $\begin{cases} ① & (p,q)\sim(p,q)\ ? \\ ② & (p,q)\sim(m,n) \Rightarrow (m,n)\sim(p,q)\ ? \\ ③ & (p,q)\sim(m,n) \text{ and } (m,n)\sim(a,b) \Rightarrow (p,q)\sim(a,b)\ ? \end{cases}$

[Hint: Need cancellation law in $\mathbb{Z}$: if $ab=ac$ and $a\neq 0$, then $b=c$.]