Lecture 1  Construction of the rational numbers.

"God created the integers, all else is the work of men." Kronecker 1886

Kronecker: Finite mathematics (Finite #s & Finite # of steps).

- Non-constructive proofs are skeptif
- Lindemann's proof that π is a nce, but proved nothing.

Weird things happen involving ∞. Ex 1: \[ 1 - 1 + 1 - 1 + 1 - 1 + \ldots \]

Example: Consider \[ \sum_{n=1}^{\infty} \frac{1}{n} \left(-1\right)^n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \ldots \]

\[ = 1 - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \left(\frac{1}{6} - \frac{1}{7}\right) - \left(\frac{1}{8} - \frac{1}{9}\right) - \ldots \]

\[ < 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} \]

But: Consider a rearranged: \[ 1 + \left(\frac{1}{2} - \frac{1}{2}\right) + \frac{1}{3} + \left(\frac{1}{4} - \frac{1}{4}\right) + \frac{1}{5} + \ldots \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots \]

\[ > \frac{5}{6} \]

How is this possible?

Morals: We need a better understanding of the fundamentals of mathematics.

- Some things which are "obvious" haven't always been, and vice versa.

Newton (1666): Infinite series = π.

But what does this mean? Forms the basis of calculus.

What does it mean for a set of it to converge?

What is a limit?

Could a sequence of rational numbers converge but not have a limit?
Can we define what it means for a sequence to converge into the notion of a limit?

**Late 1600's:** Newton, Leibniz developed calculus. The idea of a limit was there, but not made explicit.

**Early 1800's:** Fourier series invented. Laplace, Lagrange felt uneasy.

For example, we can write

\[ \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi} \sin nx. \]

Is this cheating? But it gives answers! Seemed right.

Concepts became precise.

**Cauchy 1820's**

**Weierstrass/Riemann 1850's, 60's.**

- We'll construct the rational numbers \( \mathbb{Q} \) from \( \mathbb{Z} \).

**Sets & relations**

**Definition:** A set is a collection of objects. (No repeats allowed.)

Write: \( S = \{ 1, 2, 3 \} \).

or \( S = \{ x : P(x) \text{ is true} \} \).

e.g., \( \{ x : x > 0 \} \).

**Notation:** \( x \in S \) means \( x \) is in \( S \).

\( x \notin S \) means \( x \) is not in \( S \).
Notation (cont.):

- $\emptyset$ is the empty set.
- $A \subseteq B$ means "$A$ is a subset of $B$," i.e., if $x \in A$, then $x \in B$ or $x \notin A \Rightarrow x \notin B$.

Remark: $\subseteq$ and $\subseteq$ mean the same thing! (Not like $<$, $\leq$.)

If $A \subseteq B$ and $B \not\subseteq B$, then $A$ is a proper subset of $B$.

If $A \subseteq B$ and $B \subseteq A$, then $A = B$, else $A \neq B$.

More:
- $A \cup B = \{ x : x \in A \text{ or } x \in B \}$. (union)
- $A \cap B = \{ x : x \in A \text{ and } x \in B \}$. (intersection)
- $A^c = \{ x : x \notin A \}$. (complement).
- $A \setminus B = \{ x : x \in A \text{ and } x \notin B \}$. (minus).
- $A \times B = \{ (a, b) : a \in A, \text{ and } b \in B \}$. (product)
  $\cap$ ordered pair.

Relation(s):

Def: A (binary) relation $R$ is a subset of $A \times B$.

If $(a, b) \in R$, write $a R b$.

Example: A "is an ancestor of" is a relation on $P \times P$.
L "likes" is a relation on $P \times P$.
S "is a sibling of" "is an integer.
$<$ "less than" $\mathbb{Z} \times \mathbb{Z}$.
An equivalence relation \( R \) on \( S \) is a relation on \( S \times S \) s.t.

1. Reflexive: \( aRa \)
2. Symmetric: \( aRb \Rightarrow bRa \)
3. Transitive: \( aRb \) and \( bRc \Rightarrow aRc \).

We usually write these as e.g., \( \approx, \equiv, \sim, \asymp \), etc. (\( \asymp \) is rare.)

Remark: Any function \( F : A \rightarrow B \) is a relation s.t.

if \( aFb \) and \( aFb' \) then \( b = b' \). Example: Birthdays!

We usually write \( F(a) = b \), a different notation.

Construction of \( \mathbb{Q} \) (rational numbers)

Assume \( \mathbb{Z} \), the integers, their arithmetic, and order

Question: What is \( \mathbb{Q} \)?

Fractions? Repeating decimals? What about \( 0.999\ldots \)?

\[ \{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \} \]

Not a good answer. Problem: What does \( \frac{m}{n} \) mean? ??

Motivation: \( \frac{1}{2} \) is the sum (equivalent) as \( \frac{2}{4} \)

Write: \( (1, 3) \sim (2, 6) \) equivalent ordered pairs.

Idea: These belong to some equivalence class, call it \( \frac{1}{3} \)
(What's the equivalence relation??)

Let $\mathcal{Q}$ be the set of all such equivalence classes of ordered pairs of $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$.

Goal: Have these pairs extend $\mathbb{Z}$ so that $\frac{m}{n} \in \mathcal{Q} \iff n \in \mathbb{Z}$.

Note that $\mathcal{Q} = \left\{ \frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0 \right\}$, where $\frac{m}{n}$ is an equivalence class of $(m, n)$ with relation $(p, q) \sim (m, n)$ if $mp = qn$.

Check $\sim$ is an equivalence relation on $(\mathbb{Z} \times \mathbb{Z} \setminus \{0\}) \times (\mathbb{Z} \times \mathbb{Z} \setminus \{0\})$:

\begin{align*}
1. \quad (p, q) \sim (p, q) \\
2. \quad (p, q) \sim (m, n) \Rightarrow (m, n) \sim (p, q) \\
3. \quad (p, q) \sim (m, n) \text{ and } (m, n) \sim (n, b) \Rightarrow (p, q) \sim (n, b)
\end{align*}

[Hint: Need cancellation law in $\mathbb{Z}$: if $ab = ac$ and $a \neq 0$, then $b = c$.]

HW:

1. $(p, q) \sim (p, q)$?
2. $(p, q) \sim (m, n) \Rightarrow (m, n) \sim (p, q)$?
3. $(p, q) \sim (m, n)$ and $(m, n) \sim (n, b) \Rightarrow (p, q) \sim (n, b)$?