$\boxed{\text{Lecture 5}}$ Complex numbers.

Def: Extended reals $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$

Define order: $\forall x \in \mathbb{R}$, $-\infty < x < +\infty$

arithmetic: $x + (+\infty) = +\infty$ $\qquad \dfrac{x}{+\infty} = \dfrac{x}{-\infty} = 0.$

$x + (-\infty) = -\infty$

If $x > 0$, $x \cdot (+\infty) = +\infty$, $x \cdot (-\infty) = -\infty$

$x < 0$, $x \cdot (+\infty) = -\infty$, $x \cdot (-\infty) = +\infty$

This is not a field, but just a set with some structure.

Why care? Convenient!

Every set has a supremum

Easy to distinguish between finite & infinite.

- Euclidean space $\boxed{\mathbb{R}^k := \{(x_1, \dots, x_k) : x_i \in \mathbb{R}\}}$

Define $\underbrace{(x_1, \dots, x_k)}_{\vec{x}} + \underbrace{(y_1, \dots, y_k)}_{\vec{y}} = (x_1 + y_1, \dots, x_k + y_k)$.

Multiplication doesn't really work (we can't turn it into a field)

(What would the inverse be of $(1, 0, 0, \dots, 0)$?)

<u>Scalar multiplication</u>: $c(x_1, \dots, x_k) = (cx_1, \dots, cx_k)$, $c \in \mathbb{R}$.

↖ assoc., dist., comm., etc.

$\mathbb{R}^k$ is a <u>vector space</u>.

Also, it has an <u>inner product</u> (dot product): $\vec{x} \cdot \vec{y} = \sum_{i=1}^{k} x_i y_i$.

with a <u>norm</u> (length): $\boxed{|\vec{x}| = (\vec{x} \cdot \vec{x})^{1/2}}$

This is an <u>extension</u> of $\mathbb{R}$, but we don't get a Field.

There's another way to extend $\mathbb{R}$ to get a field structure, but it only works for $\mathbb{R}^2$.

- <u>Complex number field</u>:

  $\mathbb{R}^2$ can be given a field structure.

  $$(a, b) + (c, d) = (a+c, b+d)$$
  $$(a, b) \cdot (c, d) = (ac - bd, ad + bc).$$

  Here, "zero" is $(0,0)$, and "the 1" is $(1,0)$. (Check)

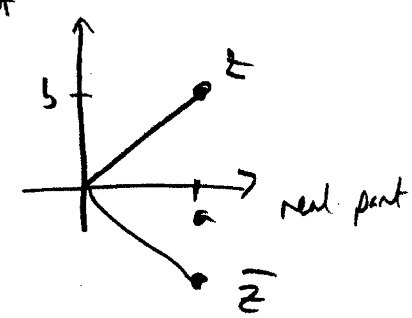
  Call this field $\mathbb{C}$. (The <u>set</u> $\mathbb{R}^2$ with $+, \cdot$ defined as above).

- $\mathbb{C}$ <u>extends</u> $\mathbb{R}$: $\{(a, 0) : a \in \mathbb{R}\}$ "behaves like $\mathbb{R}$" (a subfield <u>isomorphic</u> to $\mathbb{R}$).

<u>Note</u>: $\underbrace{(0,1)}_{\text{Call this } i} \cdot (0,1) = (-1, 0)$

<u>see</u>: $i^2 = -1$, a <u>real</u> number.

So, we write $\boxed{a+bi}$ for (a,b). imag. part ↑ Descartes



IF $z = a+bi$, $[a = Re(z),\ b = Im(z)]$

let $\bar{z} = a - bi$, the <u>conjugate</u> of z.

<u>Check</u>: $\overline{z+w} = \bar{z} + \bar{w}$.

$\overline{z \cdot w} = \bar{z} \cdot \bar{w}$.

$z + \bar{z} = 2\,Re(z)$, and $z - \bar{z} = 2i\,Im(z)$.

$z \cdot \bar{z} = a^2 + b^2$ real ≥ 0.

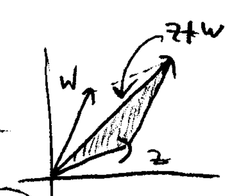
<u>Define</u> $\boxed{|z| = (z \cdot \bar{z})^{1/2}}$, same as length in $\mathbb{R}^2$

[<u>Suggests</u>: In $\mathbb{C}^k = \{(z_1, \ldots, z_k) : z_i \in \mathbb{C}\}$, the inner product $\langle\, ,\rangle$ is defined as $\langle \vec{x}, \vec{y} \rangle = \sum_{i=1}^{k} x_i \bar{y}_i$]

<u>Properties</u>: $|z| \geq 0$, $|\bar{z}| = |z|$, $|zw| = |z|\,|w|$, $Re(z) \leq |z|$.

based on $(ac-bd)^2 + (ad+bc)^2 = (a^2+b^2)(c^2+d^2)$

Also, $\boxed{|z+w| \leq |z| + |w|}$ "triangle inequality"



<u>Why</u>? $|z+w|^2 = (z+w)(\bar{z}+\bar{w}) = \underbrace{z\cdot\bar{z}}_{|z|^2} + z\cdot\bar{w} + w\cdot\bar{z} + \underbrace{w\cdot\bar{w}}_{|w|^2}$

$= |z|^2 + 2\,Re(z\bar{w}) + |w|^2$

$\leq |z|^2 + 2|z|\,|w| + |w|^2 = (|z| + |w|)^2$. ✓

This yields the desired inequality

(Rudin 1.35)

<u>Cauchy-Schwarz ineq.</u>: If $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ are complex numbers,

then $\left| \sum_{i=1}^{k} a_i \bar{b}_i \right|^2 \le \sum_{i=1}^{k} |a_i|^2 \sum_{i=1}^{k} |b_i|^2$

In $\mathbb{R}^k$: $|\vec{v} \cdot \vec{w}| \le |\vec{v}|\,|\vec{w}|$, or $|\langle \vec{v}, \vec{w} \rangle|^2 \le \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle$

In physics, this forms the basis of the Heisenberg uncertainty principle.

<u>Proof</u>: Let $\vec{a}, \vec{b} \in \mathbb{C}^n$. We need to show $|\langle \vec{a}, \vec{b} \rangle|^2 \le \langle \vec{a}, \vec{a} \rangle \langle \vec{b}, \vec{b} \rangle$

Note that $0 \le |\vec{a} - y\vec{b}|^2$ regardless of y!

(drop $\rightharpoonup$ symbol) $\quad = \langle a - yb, a - yb \rangle = \sum (a_i - y b_i)(\bar{a}_i - \bar{y} \bar{b}_i)$.

$= \langle a, a \rangle - \bar{y} \langle a, b \rangle - y \langle b, a \rangle + |y|^2 \langle b, b \rangle$

Choose $\boxed{y = \dfrac{\langle a, b \rangle}{\langle b, b \rangle}}$ $\quad = \langle a, a \rangle - \dfrac{\langle a, b \rangle \langle b, a \rangle}{\langle b, b \rangle}$

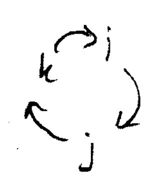
Note that $\bar{y} = \dfrac{\langle b, a \rangle}{\langle b, b \rangle}$ $\quad 0 = \langle a, a \rangle - \dfrac{|\langle a, b \rangle|^2}{\langle b, b \rangle}$

Multiplying through by $\langle b, b \rangle$ yields the desired result. $\square$

<u>Remarks</u>: Though we can't make $\mathbb{R}^k$ into a field for $k > 2$, we can "come close" in 2 cases:

<u>$n = 4$</u> $\mathbb{H} = \{ a + bi + cj + dk : a, b, c, d \in \mathbb{R} \}$ "Hamiltonians"

$i^2 = j^2 = k^2 = -1$, $ij = jk = ki = -1$. <u>Not commutative!</u>



<u>$n = 8$</u> $\mathbb{O}$ "Octonians" Complicated; <u>not associative!</u>