Lecture 6  Principle of induction.

Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, the "natural numbers."

Well-ordering property (WOP): A set $S$ is well-ordered if every non-empty subset of $S$ has a least element.

Axiom: $\mathbb{N}$ is well-ordered.

- Principle of induction (POI): Let $S \subseteq \mathbb{N}$ such that:
  
  (a) $1 \in S$,
  
  (b) If $k \in S$, then $k+1 \in S$,

  Then $S = \mathbb{N}$.

Fact: WOP $\iff$ POI.  (Both for $\mathbb{N}$)

Proof: WOP $\implies$ POI.

(by contradiction) Suppose $S$ exists with given property (a), (b) but $S \neq \mathbb{N}$. Let $A = \mathbb{N} - S \neq \emptyset$.

Then $A$ has a least element in $\mathbb{N}$, by WOP.

Then $n-1 \in S$, but then by (b), $n \in S$. \qed
Proofs by induction

Let $P(n)$ be statements indexed by $n \in \mathbb{N}$.

Plan: show $P(n)$ is true for all $n$.

We'll show:
1. $P(1)$ is true (base case),
2. If $P(k)$ is true, then $P(k+1)$ is true. (inductive step).

Then, by POI, $P(n)$ is true for all $n \in \mathbb{N}$.

What we're really doing: $S = \{ n : P(n) \text{ is true} \}$, showing $S = \mathbb{N}$.

Strong induction: use (6): If $P(1), P(2), \ldots, P(k)$ is true, then $P(k+1)$ is true. This is equivalent to POI.

Style: At start, tell the reader (proof by induction).
- Tell reader what you're doing (base case, ind. step, etc).
- Assume terms are understood.
- Remind reader of conclusion at end.

Example: Every $2^k \times 2^k$ chessboard with one square removed can be tiled by $\square$.

Proof: (by induction on $n$).

- For base case, see $\square$, the statement holds.
- For inductive step, we can assume any $2^{k+1} \times 2^{k+1}$ board with a square removed can be tiled.
So consider a $2^{k+1} \times 2^{k+1}$ board with one square removed. Can divide the board into 4 parts, 3 full $2^k \times 2^k$ boards.

The part with a "hole" can be tiled by TLEOP.

We can remove a tile from the other 3, see figure.

So $2^{k+1} \times 2^{k+1}$ board can be tiled.

By P01, the statement holds.

Example:

Theorem: Prove $S_n = 1 + 3 + 5 + \ldots + (2n-1)$ is a perfect square.

Proof: Base case: $(n = 1)$ holds because $1 = 1^2$.

Inductive step: Assume $S_n$ is a square, say $S_n = k^2$.

Want to show: $S_{n+1}$ is a square:

$$S_{n+1} = 1 + 3 + 5 + \ldots + (2n-1) + 2n+1$$

$$= S_n + (2n+1) \quad \text{strengthen!}$$

$$= n^2 + 2n + 1$$

$$= (n+1)^2.$$ 

We've proven that $S_n = n^2$.
Theorem: All tigers have the same color.

Proof (by induction on # of horses).

For base case: case of 1 horse, statement holds.

For inductive hyp., assume it holds for k horses.

Consider a set of k+1 horses, \( S = \{ h_1, \ldots, h_{k+1} \} \).

Define \( S' = \{ h_1, \ldots, h_k \} \) and \( S'' = \{ h_k, \ldots, h_{k+1} \} \).

By IH, these two n-element sets all contain horses of the same color. And they overlap! \( h_2 \) is in both. So every horse in \( S \) has the same color! \( \square \)

Flaw: The inductive step \(( h_2 \in S' \cap S'' ) \) fails for \( n = 2 \).