MTHSc 453

Lecture 6 — Principle of induction.

Let $\mathbb{N} = \{1, 2, 3, \dots\}$, the "natural numbers."

<u>Well-ordering property</u> (WOP): A set S is <u>well-ordered</u> if every nonempty subset of S has a least element.

<u>Axiom</u>: $\mathbb{N}$ is well-ordered.

- <u>Principle of induction</u> (POI) Let $S \subset \mathbb{N}$ such that:

  (a) $1 \in S$,

  (b) If $k \in S$, then $k+1 \in S$,

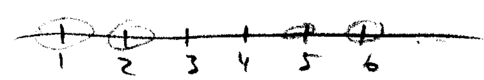
  Then $S = \mathbb{N}$.

<u>Fact</u>: WOP $\Leftrightarrow$ POI. (Both for $\mathbb{N}$)

<u>Proof</u>: WOP $\Rightarrow$ POI.

(by contradiction). Suppose S exists with given properties (a), (b), but $S \neq \mathbb{N}$.

Let $A = \mathbb{N} - S \neq \emptyset$.



Then A has a least element $n > 1$, by WOP.

Then $n-1 \in S$, but then by (b), $n \in S$. ↯

$\square$

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Proofs by induction

Let $P(n)$ be statements indexed by $n \in \mathbb{N}$.

Idea: Show $P(n)$ is true for all n.

We'll show: (a) $P(1)$ is true (base case),

(b) If $\underbrace{P(k) \text{ is true}}_{\text{induction hypothesis.}}$, then $P(k+1)$ is true. (inductive step).

Then, by POI, $P(n)$ is true for all $n \in \mathbb{N}$.

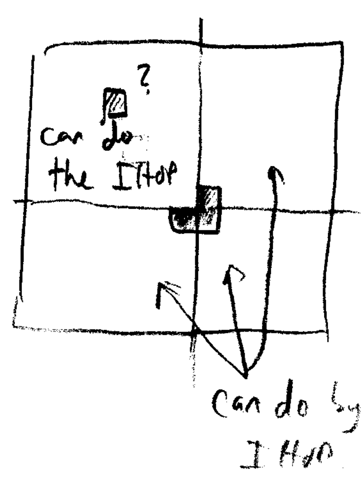
What we're really doing: $S = \{n : P(n) \text{ is true}\}$, showing $S = \mathbb{N}$.

Strong induction: Use (b'): If $P(1), P(2), \ldots, P(k)$ is true, then $P(k+1)$ is true. This is equivalent to POI.

Style: At start, tell the reader (proof by induction).

- Tell reader what you're doing (base case, ind. step, etc).
- Assume terms are understood.
- Remind reader of conclusion at end.

Example: Every $2^n \times 2^n$ chessboard with one square removed can be tiled by 



Proof: (by induction on n).

- For base case, see , the statement holds.
- For inductive step, we can assume any $2^k \times 2^k$ board with a square removed can be tiled.

So consider a $2^{k+1} \times 2^{k+1}$ board with one square removed.

Can divide the board into 4 parts, 3 full $2^k \times 2^k$ boards.

The part with a "hole" can be tiled by IHOP.

We can remove a tile from the other 3; see figure.

So $2^{k+1} \times 2^{k+1}$ board can be tiled.

By POI, the statement holds. $\square$

Examples

Theorem: Prove $S_n = 1+3+5+\cdots+(2n-1)$ is a perfect square.

Proof: Base case: $(n=1)$ Holds because $1 = 1^2$.

Inductive step: Assume S_n is a square, say $S_n = k^2$.

Want to show: S_{n+1} is a square:

Strengthen: n^2

$$
\begin{aligned}
S_{n+1} &= 1+3+5+\cdots+(2n-1) + 2n+1 \\
&= S_n + (2n+1) \\
&= n^2 + 2n + 1 \\
&= (n+1)^2.
\end{aligned}
$$

strengthen!

We've proven that $S_n = n^2$.

$\square$

(4)

Theorem: All tigers have the same color.

Proof: (by induction on # of horses).

For base case: case of one horse. Statement holds.

For ind hyp, assume it holds for k horses.

Consider a set of k+1 horses, $S = \{h_1, .., h_{k+1}\}$.

Define $S' = \{h_1, .., h_k\}$ and $S'' = \{h_2, ..., h_{k+1}\}$.

By IHOP, these two n-element sets all contain horses of the same color. And they overlap: h_2 is in both. So every horse in S has the same color! $\square$

Flaw: The inductive step ($h_2 \in S' \cap S''$) fails for $n = 2$.