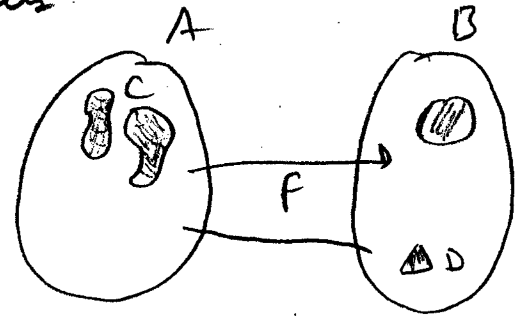
MTHSc 453

Lecture 7 & 8 Countable & uncountable sets



How to count:

- Recall: $\underset{\text{domain}}{f:A} \to \underset{\text{codomain}}{B}$
  "maps" $x \mapsto f(x)$.

If $C \subset A$, $D \subset B$, define $f(C) = \{f(x) : x \in C\}$, the image of C

and $f^{-1}(D) = \{x \in A : f(x) \in D\}$, the inverse image, or preimage of D.

Note that f^{-1} need not be a function.

Def: • $f(A)$ is the range of f. When $f(A) = B$, say f is onto (or surjective). Write $f : A \twoheadrightarrow B$

- when $f(x) = f(y)$ implies $x = y$, say f is 1-1 (or injective) write $f : A \hookrightarrow B$.

- When f is 1-1 & onto, call f a bijection, and we say A & B are in 1-1 correspondence. Write $A \sim B$.

Elementary counting: Use $A = J_n = \{1, 2, \ldots, n\}$.

Ex: $\{1, a, \ , \emptyset\} = \boxed{A}$. Say $|A| = 4$.

$$\begin{array}{ccccc} 1 & a & \ & \emptyset & A \\ \uparrow & \uparrow & \uparrow & \uparrow & \updownarrow \\ 3 & 1 & 4 & 2 & J_4 \end{array}$$

Def: Call A *finite* if $A \sim J_n$ for some n. [Call $J_0 = \emptyset$].

Otherwise, call A *infinite*.

Def: Call A *countable* if $A \sim \mathbb{N}$. [Sometimes, we include finite sets too]

Ex: • $\mathbb{N}$ is countable: use $f: \mathbb{N} \to \mathbb{N}$, $f(x) = x$.

• A sequence $x_1, x_2, x_3, \ldots$ of distinct terms is countable:

use $f: \mathbb{N} \to \{x_n\}$ $\quad f(n) = x_n$.

* **Moral:** A set that can be "listed" in a sequence is countable!

Ex: • $\{2, 3, 4, 5, \ldots\}$ is countable, use $f(n) = n+1$

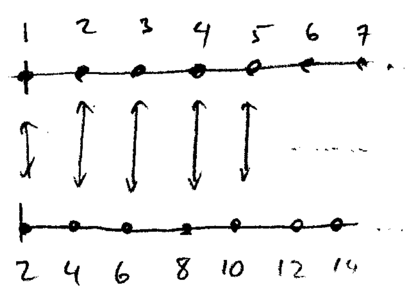
• $\{1, 2, 3, \ldots, k-1, k+1, k+2, \ldots\}$ is countable: use $f(n) = \begin{cases} n & n < k \\ n+1 & n \geq k \end{cases}$

Theorem: $\mathbb{N}$ is infinite.

Proof (Sketch) Induction. I'll skip this

Ex: $\mathbb{N} \sim 2\mathbb{N} = \{2, 4, 6, 8, 10, \ldots\}$. Use $f(n) = 2n$.

$\mathbb{N}$, $2\mathbb{N}$ have the same cardinality



Ex: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is countable

$\ldots 7 \ 5 \ 3 \ 1 \ 2 \ 4 \ 6 \ldots$

Thm: Every infinite subset E of a countable set A is countable.

Proof: (sketch). $A = \{x_1, x_2, x_3, x_4, x_5, \dots\}$

in E

let $n_1 = \inf\{i : x_i \in E\}$

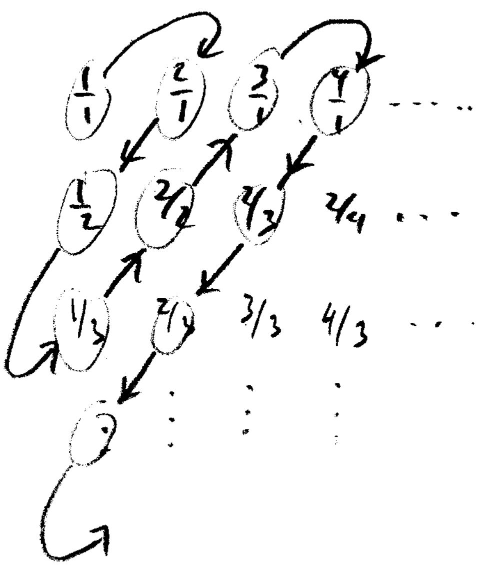
$n_2 = \inf\{i : x_i \in E,\ i > n_1\}$

$\vdots$

$n_k = \inf\{i : x_i \in E,\ i > n_{k-1}\}$. Then $E = \{x_{i_1}, x_{i_2}, x_{i_3}, \dots\}$. $\square$

Thm: $\mathbb{Q}$ is countable.

Proof



Remarks:
- This shows that ordered pairs, (m, n), $m, n \in \mathbb{Z}$ are countable.
- $\mathbb{Q}^+$ is a subset of this
- It's easy to extend this to all of $\mathbb{Q}$.

Thm: If A is countable, then $A \times A$ is countable.

Theorem: $\mathbb{R}$ is not countable.

Proof: (by contradiction). Suppose $\exists$ bijection $f : \mathbb{N} \to \mathbb{R}$.

Consider the list $f(1), f(2), f(3), \dots$

$1 \xmapsto{f} 0.1234567890\ldots$ [Disallow infinite strings of 9]

$2 \longmapsto 0.3234170092\ldots$

$3 \longmapsto 0.3141592653\ldots$

$4 \longmapsto 0.9871243882$

$5 \longmapsto 0.8767234100$

increase each digit by $\begin{cases} 1 & \text{if it's } 0\text{–}7 \\ -1 & \text{if it's } 8\text{–}9. \end{cases}$

Construct a new number $x^* = 0.23523\ldots$

Note that $x^* \neq f(1)$ (differs in 1st digit)

$x^* \neq f(2)$ (differs in 2nd digit)

$x^* \neq f(3)$ (differs in 3rd digit).

so x^* cannot be $f(n)$ for any n, contradicting the assumption that f is surjective. $\square$

Sets of different cardinalities: $J_0, J_1, \ldots, J_n, \ldots, \mathbb{N}, \mathbb{R}, ?\ldots$

Remark: The question of if there is a set A with cardinality $|\mathbb{N}| < |A| < |\mathbb{R}|$ is undecidable, i.e., it is neither true nor false! This is the continuum hypothesis.

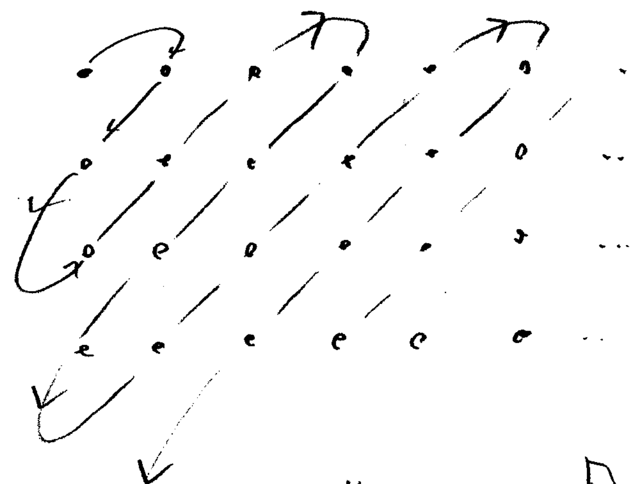
Theorem: The countable union of countable sets is countable.

Proof: Say each $A_1, A_2, A_3, \ldots$ is countable

Then $A_1 = \{a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, \ldots\}$

$A_2 = \{a_{21}, a_{22}, a_{23}, a_{24}, a_{25}, \ldots\}$

Apply Cantor's diagonal argument:



$\square$

Remark: We can replace "countable" with "at most countable"!

Notation: Use $\bigcup_{\alpha \in J} A_\alpha$ for possibly uncountable collection J.

Ex: • The set of computer programs is countable!

• The set of all possible outputs to computer programs is thus (at most) countable.

But the set $\mathbb{R}$ is uncountable!

* Thus, there are real numbers that are not "computable"! (In fact, "most"!)

i.e., input: $k \in \mathbb{N}$, output k decimal places.

• Given a set A, the power set 2^A is the set of all subsets of A

Example: $A = \{1, a, 8^{\circ\circ}\}$ $\quad 2^A = \{\emptyset, \{1\}, \{a\}, \{8^{\circ\circ}\}, \{1, 8^{\circ\circ}\}, \dots\}$

call this B.

Encode B as 101 $\quad 1 = $ yes, $a = $ no, $8^{\circ\circ} = $ yes

There is a bijection $2^A \longrightarrow$ binary strings of length $2^{|A|}$.

<u>Cantor's Theorem</u> (diagonalization argument, 1891):

For any set A, we have $\boxed{A \nsim 2^A}$

<u>Proof</u>: (by contradiction).

Suppose $\exists$ bijection $f : A \hookrightarrow\!\!\!\rightarrow 2^A$

$$a \longmapsto f(a) \subset A.$$

$$\text{e.g., } a \longmapsto \{1, 🐾, a\}.$$

<u>Idea</u>:

$a \in A$	$f(a) \in 2^A$	Is $a \in f(a)$?	
$\alpha \longrightarrow$	$\{A, y, 🐾, \alpha, \{a, b\}\}$	$a \in f(a)$	✓
🐾 $\longrightarrow$	$\{*, \phi, 1, 3, \ldots\}$	🐾 $\notin f($🐾$)$	×
$* \longrightarrow$	$\{*, \phi, y, 🐾, a\}$	$* \in f(*)$	✓

Want subset $B \in 2^A$ <u>not</u> in image of f.

Let $\boxed{B = \{a : a \notin f(a)\}}$

So, if $B = f(x)$ for some $x \in A$, consider x. $\left(x \xmapsto{f} B.\right)$

* <u>Is $x \in B$?</u> <u>No</u>, because then $x \notin f(x) = B$, contradiction

Then $x \notin B$, then $x \in f(x) \setminus B$, so $x \in B$, Contradiction

So $B \neq f(x)$ for any $x \in A$, thus such a bijection can't exist. $\square$

Consider the set $F_A := \{ f : A \to \{0,1\} \}$, where A is any set.

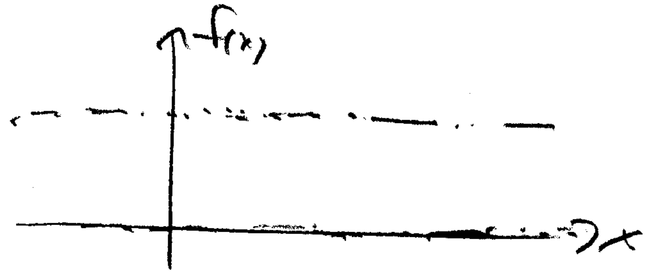
Claim: $2^A \sim F_A$.

Proof (sketch). There is a bijection

$$2^A \longrightarrow F_A$$

$$S \longmapsto f(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S. \end{cases}$$

Example: If $A = \mathbb{R}$, then F_A is the set of all real-valued functions $\mathbb{R} \to \{0,1\}$.



There are $2^{\mathbb{R}}$ such functions!

Questions:

- How big is the set $\{ f : \mathbb{R} \to \mathbb{R} \}$?
- How big is the set $\{ f : \mathbb{R} \to \mathbb{R},\ f \text{ is continuous} \}$?

We'll discuss these later...