$\boxed{\text{lecture } 8 \,\&\, 9}$ Metric spaces

Question: How to measure distance? ... in $\mathbb{R}^n$

in genome sequences?

Def. A set X is a metric space if $\exists$ a metric

$d : X \times X \longrightarrow \mathbb{R}$ such that $\forall\, p, q \in X$:

(a) $d(p,q) \geq 0$ (and $=0$ iff $p=q$), "nonnegative"

(b) $d(p,q) = d(q,p)$, "symmetric"

(c) $d(p,q) \leq d(p,r) + d(r,q)$. "triangle inequality"

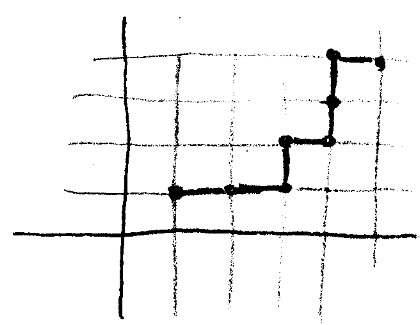
We usually write a metric space as $(\underset{\text{set}}{X}, \underset{\text{metric}}{d})$.

Examples:

- $X = \mathbb{R}$, $d(x,y) = |x-y|$
- $X = \mathbb{R}^k$, $d(\vec{x}, \vec{y}) = \sqrt{(x_1-y_1)^2 + \cdots + (x_k, y_k)^2}$ "usual metric on $\mathbb{R}^k$"
- $X = \mathbb{R}^k$, $d(\vec{x}, \vec{y}) = \sum_{i=1}^{k} |x_i - y_i|$.

"staircase", or "taxicab" metric.

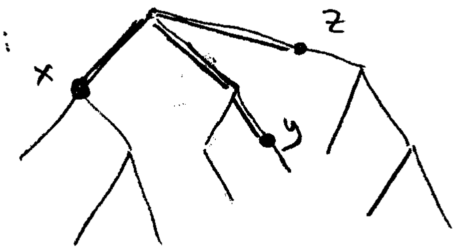
In $\mathbb{R}^2$:



Examples. (cont)

- $X = \mathbb{R}^k$, $d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$ "discrete metric"

- X = fixed tree:



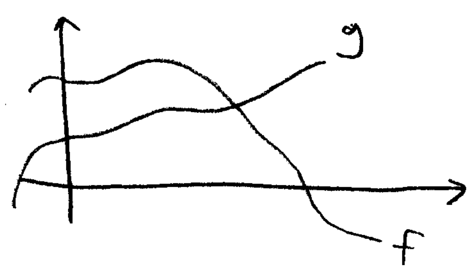
$d(x,y)$ = length of shortest path from x to y.

- $X = \{$genome sequences of length $n\}$, $d(\vec{x}, \vec{y})$ = # places where they differ.

  e.g., $\vec{x}$: GATTACA
  $\vec{y}$: AGATCAT

  [Is this a metric?]

- $\mathcal{C}([a,b])$: space of continuous functions $[a,b] \longrightarrow \mathbb{R}$



How to define distance?

There are several ways:

* $d(f,g) = \int_a^b |f-g| \, dx$ "L^1-norm"

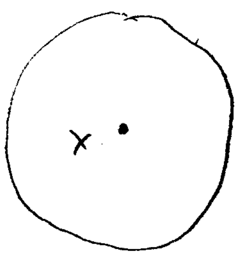
or * $d(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$ "sup-norm" or "L^∞-norm"

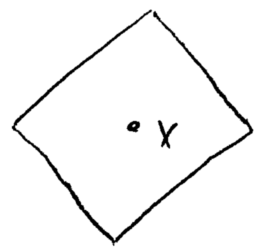
or * $d(f,g) = \left(\int_a^b (f-g)^2 \, dx \right)^{1/2}$ "L^2-norm"

<u>Open ball</u> (or "neighborhood") $N_r(x) = \{ y \in X : d(x,y) < r$

<u>Closed ball</u>: $\overline{N_r(x)} = \{ y \in X : d(x,y) \leq r \}$.

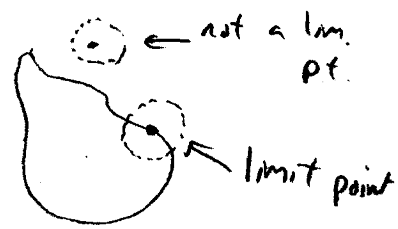
<u>In</u> $\mathbb{R}^2$: (usual metric) (staircase metric)





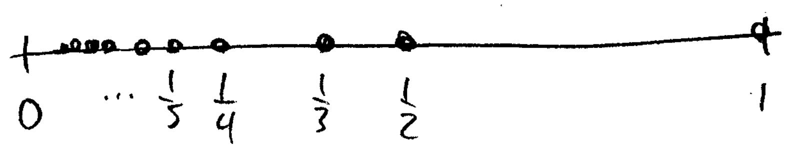
<u>Think</u>: What about the discrete metric?

* <u>Def</u>. Say $p \in X$ is a <u>limit point</u> of E if <u>every</u> neighborhood (open ball) of p contains a point $q \neq p$ such that $q \in E$.



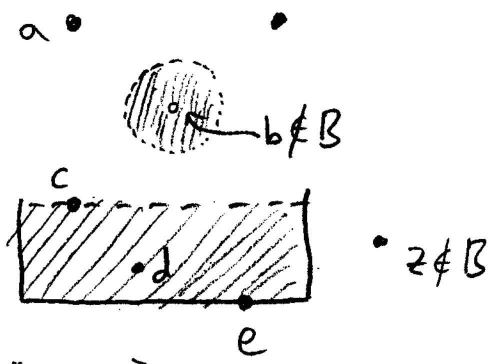
<u>Ex</u>: Consider the set $G = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots\} = \{\frac{1}{n} : n \in \mathbb{N}\}$.

The point $0 \notin G$, but 0 <u>is</u> a limit point of G.



(4)

<u>Ex</u>: In $\mathbb{R}^2$, consider B:



<u>Limit points</u>: b, c, d, e [though $b, c \notin B$]

<u>Not</u> limit points: a, z [a is "isolated", $z \notin B$]

<u>Remark</u>: A point $p \in X$ is <u>not</u> a limit point of E if $\exists$ nbhd N of p s.t. N does <u>not</u> contain any <u>other</u> point of E.

<u>Def</u>. A point $p \in X$ is an <u>isolated point</u> of E if $p \in E$, but p is <u>not</u> a limit point of E.

<u>Ex</u>: All points of G are isolated.

There are two isolated points of B.

<u>Def</u>. A point $p \in X$ is an <u>interior point</u> of E if $\exists$ nbhd N of p s.t. $N \subset E$.

[So p is <u>not</u> an interior point if $\forall$ nbhds N of p, N contains some point <u>not</u> in E.]

<u>Ex</u>: • G has <u>no</u> interior points

• In B, d is an interior point, but a, b, c, e are <u>not</u>.

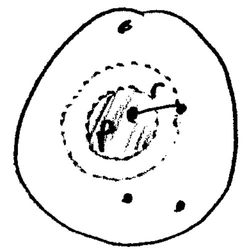
• In $(\mathbb{R}, \text{disc})$, if $E = \{x\}$, the x is <u>both</u> an interior point <u>and</u> an isolated point!

<u>Example</u>: • In $\mathbb{R}$, consider the sets ϕ, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$. [Do in small groups]

What are the
* limit points?
* isolated points?
* interior points?

• Same question, but for $(\mathbb{R}, \text{discrete})$.

<u>Theorem</u>: If p is a <u>limit point</u> of E, then every <u>nbhd</u> of p contains infinitely many points of E.



<u>Proof</u>: (contrapositive.)

Suppose $\exists$ nbhd N of p with only finitely many points of E: $e_1, \dots, e_n$.

Let $r = \min_{1 \le i \le n} \{d(p, e_i)\}$, which exists since $\{e_1, \dots, e_n\}$ is finite.

Consider $N_{r/2}(p)$, which has no point of E.

Thus, p is not a limit point of E. $\square$