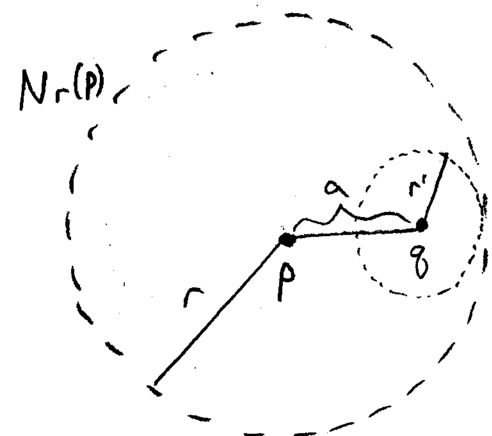
MTHSc 453

## Lecture 10 Open & closed sets.

Throughout, let $(X, d)$ be a metric space.

**Def:** A set $U \subset X$ is <u>open</u> if every point of $U$ is an <u>interior point</u> of $U$. A set $V \subset X$ is <u>closed</u> if $V$ contains all of its limit points.

**Examples:** [Motivation: The set of all non-colinear triples in $\mathbb{R}^2$; open!]




**Lemma:** Neighborhoods are <u>open</u>.

**Proof:** Consider a neighborhood $N_r(p)$.

Let $a = d(p,q) < r$, and put $r' = r - a$.

Claim: $N_{r'}(q) \subset N_r(p)$.

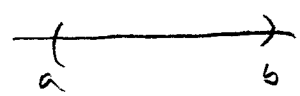
Check: Take any $x \in N_{r'}(q)$.

Need to show $x \in N_r(p)$.

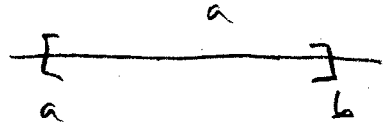
Indeed, $d(p,x) \leq d(p,q) + d(q,x)$ ← triangle inequality

$$< a + r' = a + (r-a) = r. \checkmark \qquad \square$$

- In $\mathbb{R}$, "open intervals" $(a,b)$ are open.




"Closed intervals" $[a,b]$ are closed



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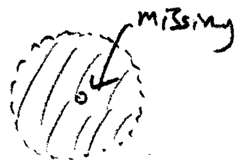
A singleton set $\{p\}$ is closed (in $\mathbb{R}$).

"Half-open" intervals $(a,b]$ and $[a,b)$ are neither open nor closed.

What about the sets $\emptyset$ and $\mathbb{R}$? (These are "clopen"!)

- In $\mathbb{R}^2$:  , 

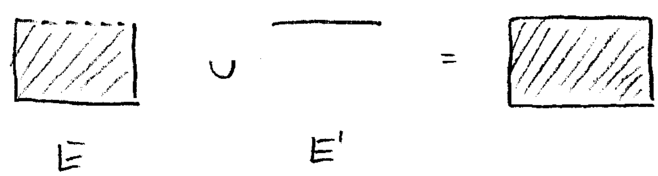
are open.

The closed neighborhood $\overline{N_r(x)} = \{p \in X : d(p,x) \leq r\}$ is closed.

Recall B:  The "mouth" of B is not closed.

Can we close it??

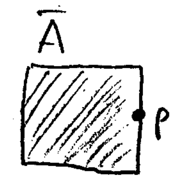
Def: Let E be the set of limit points of E. The closure of E is $E \cup E'$.

Ex: 


Theorem: For any set A, the set $\overline{A}$ is closed.

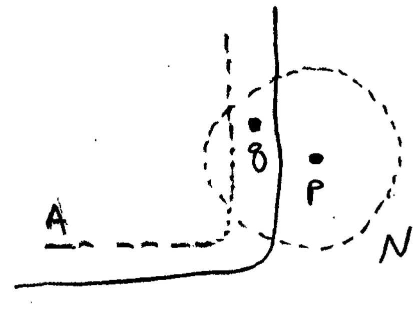
Proof: Consider p, a limit point of $\overline{A}$.

Want to show: $p \in \overline{A}$.




Consider a nbhd of $p$. If $p \in A$, the result is trivial, so assume $p \notin A$.

We'll show $N$ contains a point of $A$.




Since $p$ is a limit point of $\bar{A}$, $N$ contains a point $q \in \bar{A}$.

If $q \in A$, we're done. (We've found our desired point.)

If $q \notin A$, then $q \in A'$, so every neighborhood of $q$ contains a point in $A$.

But $N$ is open, and $q \in N$!

Thus, $\exists$ nbhd $N' \subset N$ containing $q$.

Since $q$ is a limit point, $N \cap A \neq \emptyset$, and thus $N \cap A \neq \emptyset$.

Therefore, $p$ is a limit point of $\bar{A}$, hence $\bar{A}$ is closed. $\square$

<u>Theorem</u>: $E$ is closed iff $E = \bar{E}$.

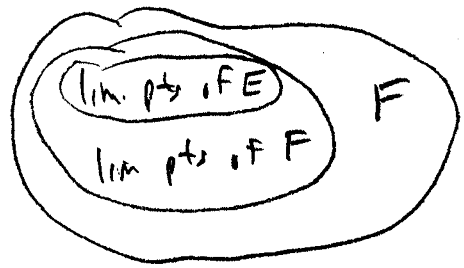
<u>Proof</u>: ($\Rightarrow$) If $E$ is closed, then $E' \subset E$ so $\bar{E} = E \cup E' = E$. ✓

($\Leftarrow$) If $E = \bar{E}$, then $E$ contains its limit points.

<u>Theorem</u>: If $E \subset F$, where $F$ is closed. Then $\bar{E} \subset F$.

[That is, $\bar{E}$ is the "smallest" closed set containing $E$.]

Proof: If $p$ is a limit point of $E$, then it's a limit point of $F$. But $F$ contains all of its limit points, so it contains all limit points of $E$. ex. $\square$

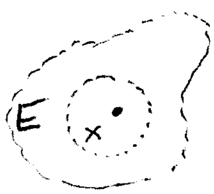



**Relationship between open & closed sets.**

Theorem: $E$ is open $\iff$ $E^c$ is closed.

[Here, $E^c = X \setminus E$; the "complement" of $E$.]

Proof: $E$ open $\iff$ any point of $x \in E$ is an interior point.




$\iff \forall x \in E$, $\exists$ nbhd $N$ of $x$ s.t. $N \cap E^c = \emptyset$.

$\iff \forall x \in E$, $x$ is not a limit point of $E^c$.

$\iff E^c$ contains all of its limit points. $\square$

Unions & Intersections:

Lemma: $\{E_\alpha\}$ collection of sets. Then $\boxed{\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha E_\alpha^c.}$

Proof. $x \in LHS \iff x \notin$ any $E_\alpha \iff x \in$ all $E_\alpha^c \iff x \in \bigcap_\alpha E_\alpha^c$. $\square$

Question: What can we say about when unions & intersections preserve open & closed sets.

Theorem.

(a) Arbitrary union of open sets is open.

(b) " intersection " closed " " closed.

(c) Finite intersection " open " " open

(d) " union " closed " " closed.

First, let's see why (c) & (d) can't be made "infinite".

Examples: $\bigcap_{n=1}^{\infty} \underset{\text{open}}{\underbrace{(-\tfrac{1}{n}, \tfrac{1}{n})}} = \underset{\text{closed.}}{\underbrace{\{0\}}}$ $\qquad \bigcup_{n=1}^{\infty} \underset{\text{closed}}{\underbrace{[-1+\tfrac{1}{2^n}, 1-\tfrac{1}{2^n}]}} = \underset{\text{open.}}{\underbrace{(-1, 1)}}$

Proof (of theorem).

(a) Pick $x \in \bigcup_\alpha U_\alpha$, each $U_\alpha$ is open.

So $x \in N_\alpha$ for some $N_\alpha \subset U_\alpha \subset \bigcup_\alpha U_\alpha$. ✓

(b) Say $B_\alpha$ is closed. Then $U_\alpha := B_\alpha^c$ is open.

So $\bigcup_\alpha B_\alpha^c$ is open $\Rightarrow \left(\bigcup_\alpha U_\alpha\right)^c \underset{\text{by lemma}}{=} \bigcap_\alpha U_\alpha^c = \bigcap_\alpha B_\alpha$ is closed. ✓

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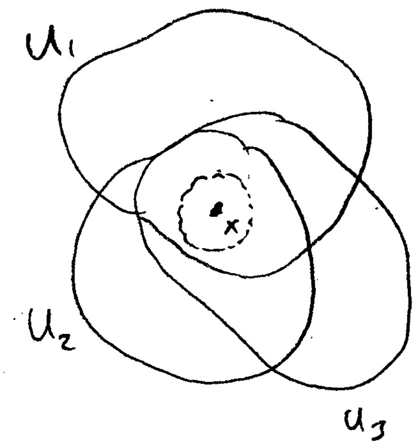
(c) $\exists\, N_{r_i}(x)$ for each $U_i$ s.t. $N_{r_i}(x) \subset U_i$.




Take $r = \min\{r_1, \dots, r_n\}$.

We have $N_{r_i}(x) \subset U_i$ for each $i$,

thus $N_{r_i}(x) \subset \bigcap_{i=1}^{n} U_i$.

[Think: Why does this fail for infinitely many $U_i$'s?]

(d) Exercise. (Several different ways to do this!) $\square$

<u>Def</u>: $E$ is <u>dense</u> in a metric space $X$ if every point of $X$ is either <u>in $E$</u> or a <u>limit point of $E$</u>.

<u>Ex</u>: $\mathbb{Q}$ is <u>dense</u> in $\mathbb{R}$.

* <u>Equiv. def'ns</u>:
- $E$ is <u>dense</u> iff $\overline{E} = X$.
- $E$ is dense if every open set of $X$ contains a point of $E$.

<u>Motivation</u>: (in related fields)

- The set of polynomials is dense in the space of analytic functions
- The set of sines & cosines is dense in the space of periodic functions.