MTHSc 453

Lecture 11 & 12 Compactness

Motivation: Analysis is interesting because of infinite sets.

Finite sets are easy to deal with. For example:

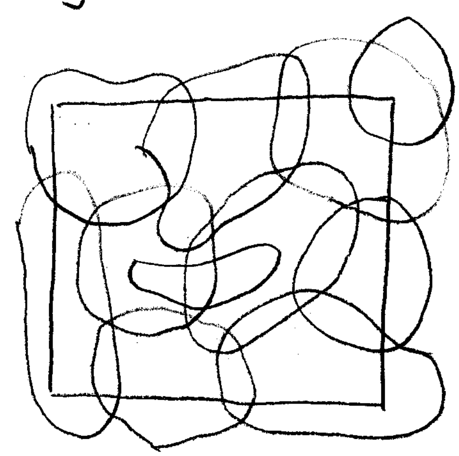
* They are bounded
* They are closed
* If $|S| < \infty$, then $\sup S = \max S$, $\inf S = \min S$.

Compact sets are the "next best thing to being finite."

Def: • An (open) cover of E in X is a collection of open sets, $\{G_\alpha\}$, whose union contains E.

• A subcover $\{G_\alpha\}$ is a subcollection $\{G_{\alpha_\beta}\}$ that still covers E.

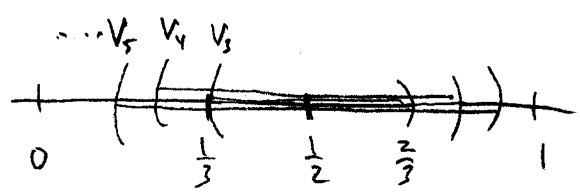
E



Example: In $\mathbb{R}$, $[\frac{1}{2}, 1)$ has cover $\{V_n\}_{n=3}^{\infty}$ where $V_n = (\frac{1}{n}, 1-\frac{1}{n})$.

There are many such covers.

For example, $\{(0,2)\}$ is a cover too.



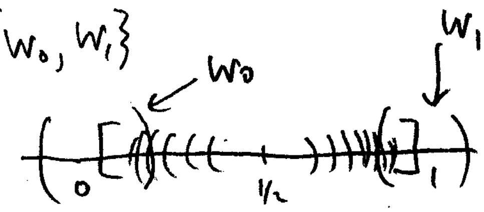
Another cover: $\{W_x\}_{x\in[\frac{1}{2},1)}$, $W_x : N_{1/10}(x)$.

<u>Question</u>: Given a <u>cover</u>, do we need all the sets to still cover?

e.g., $\{V_n\}_{n=3}^{\infty}$ has a subcover $\{V_n\}_{n=453}^{\infty}$.

$\{W_x\}$ has a <u>finite</u> subcover: $\{W_{5/10}, W_{6/10}, \dots, W_{9/10}\}$.

<u>Example</u>: $[0,1]$ in $\mathbb{R}$ has cover: $\{V_n\} \cup \{W_0, W_1\}$



<u>Key remark</u>: This cover of $[0,1]$

<u>finite subcover</u>: $\{W_0, W_1, V_{11}\}$.

$W_0 = (-\frac{1}{10}, \frac{1}{10})$, $W_1 = (\frac{9}{10}, \frac{11}{10})$.

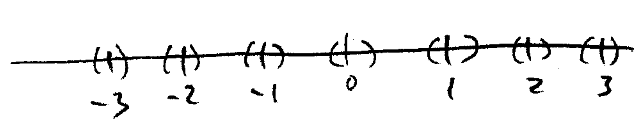
However, the cover $\{V_n\}_{n=3}^{\infty}$ does <u>not</u> have a finite subcover!

<u>Def</u>: A set K is <u>compact</u> (in X) if every <u>open cover</u> of K contains a <u>finite subcover</u>.

[So K is not compact if $\exists$ an open cover with no finite subcover.]

<u>Example</u>: • $[\frac{1}{2}, 1)$ is <u>not</u> compact (see $\{V_n\}$).

• $\mathbb{Z}$ (in $\mathbb{R}$) is <u>not</u> compact:

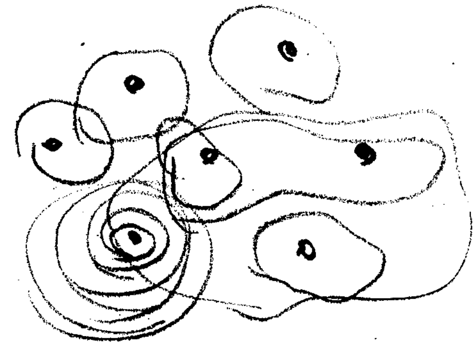


<u>Question</u>: Is $[0,1]$ compact? We don't know yet — we'd have to check <u>every</u> open cover! (Or prove a theorem.

<u>Theorem</u>: Finite sets are compact.

<u>Proof</u>: Consider an open cover $\{G_\alpha\}$,

covering $x_1, \dots, x_n$.



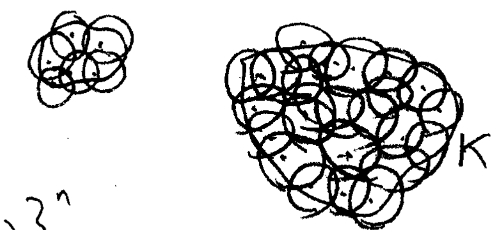
For each x_i, choose one G_{α_i} such

that $x_i \in G_{\alpha_i}$. Clearly, $\{G_{\alpha_1}, \dots, G_{\alpha_n}\}$ covers $\{x_1, \dots, x_n\}$. $\square$

<u>Theorem</u>: <u>Compact</u> sets are <u>bounded</u>. (<u>Think</u>: How to define "bounded"?)

<u>Proof</u>: Let K be compact.

Let $B(x) = N_1(x)$, the ball of radius 1.

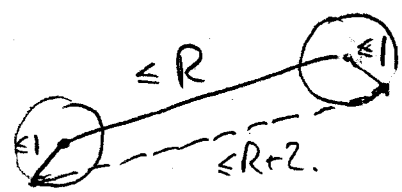


By compactness of K, $\exists$ finite subcover $\{B(x_i)\}_{i=1}^n$.

Let $R = \max_{i,j \le n} \{ d(x_i, x_j) \}$, which exists since $x_1, \dots, x_n$ is finite.

Then $N_{R+2}(x)$ contains all points in K,

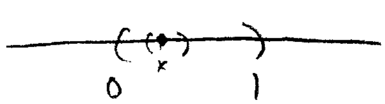
by the triangle inequality. $\square$

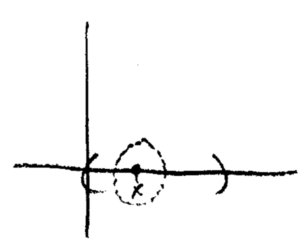


<u>Remark</u>: The property of a set being <u>open</u> depend on what

space in which it lives.

<u>Ex</u>: $(0,1)$ is <u>open</u> in $\mathbb{R}$, but <u>not open</u> in $\mathbb{R}^2$:

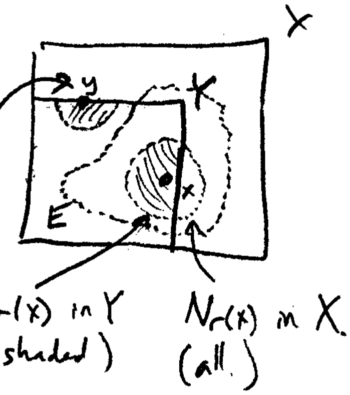




Moral (preview) Compactness does not depend on the ambient space.

"Relative" open sets: Let Y be a metric space.

If $Y \subset X$, then Y "inherits" a metric from X.



Def: A set U is open in Y (or "relative" to Y) if every point of U is an $\underbrace{\text{interior point}}_{\leftarrow \text{ using nbhd in } X}$ of U.

Theorem: Let $E \subset Y \subset X$.

Then E is open in $Y \iff E = Y \cap G$ for some open set G in X.

Proof (sketch – details left as an exercise).

($\Leftarrow$) Use: If $N_r(x) \subset G$, then $N_r(x) \cap Y$ is nbhd of x in Y, & in E. ✓

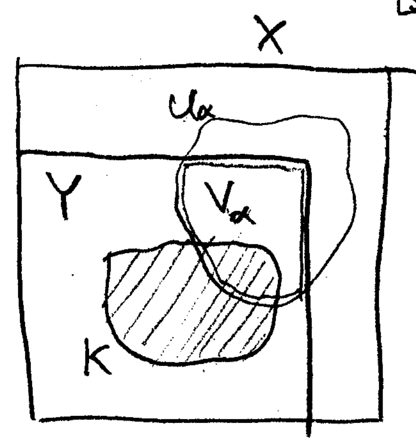
($\Rightarrow$) If E is open in Y, then every pt. x has $N_{r_x}(x) \subset Y \cap E$.

Then $G := \bigcup_{x \in E} N_{r_x}(x)$ is open in X. ✓ $\square$

Theorem: If $K \subset Y \subset X$, then K compact in $Y \iff K$ compact in X.

Proof: ($\Rightarrow$) (Assume K compact in Y.)

Consider an open cover $\{U_\alpha\}$ of K in X.



Let $V_\alpha = U_\alpha \cap Y$. Then $\{V_\alpha\}$ covers K in Y.

By compactness of K in Y, $\exists$ finite subcover $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$ of K in Y.

The corresponding subcover $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a finite subcover of $\{U_\alpha\}$. ✓

($\Leftarrow$) Consider an open cover $\{V_\alpha\}$ of K in Y.

By previous theorem, $\exists\ U_\alpha$ s.t. $U_\alpha \cap Y = V_\alpha$.

Now, $\{U_\alpha\}$ covers K in X, so $\exists$ finite subcover $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ of K in Y.

Then, $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$ is a finite subcover of $\{V_\alpha\}$. ✓ $\square$

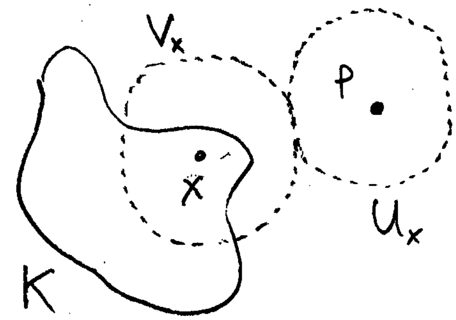
Moral: Compactness is an intrinsic property of a set (and metric) rather than relative to a larger set.

Theorem: Compact sets are closed.

Proof: Let K be compact.

We'll show that K^c is open.

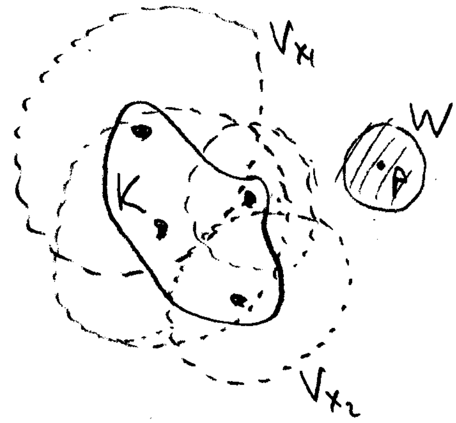
Take $p \in K^c$. Goal: Show p is an interior point of K^c (i.e., not a limit point of K.)



For each $x \in K$, let $V_x = N_{r/2}(x)$,

and $U_x = N_{r/2}(p)$, where $r = d(p,x)$.

Notice that $\{V_x \,;\, x \in K\}$ is an <u>open cover</u> of K, so by compactness, $\exists$ <u>finite subcover</u> $\{V_{x_1}, \dots, V_{x_n}\}$ of K.



Consider the set $W = U_{x_1} \cap \dots \cap U_{x_n}$.

Clearly, $p \in W$, and W is disjoint from our finite subcover of K.

Therefore, $W \cap K = \emptyset$, i.e., $W \subset K^c$.

Hence, K^c is <u>open</u>, as desired. $\square$

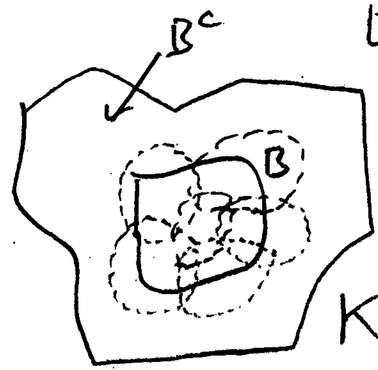
<u>Cor</u>: If $E \subset X$ is not closed, then E is not compact. $\square$

<u>Remarks</u>:
- $(0,1)$ is not compact.
- Closed sets need not be compact (e.g., $\mathbb{Z}$ or $\mathbb{R}$ in $\mathbb{R}$).

<u>Theorem</u>: Closed subsets of compact sets are compact.

<u>Proof</u>: Let $B \subset K$, where B is closed and K compact.

Let $\{U_\alpha\}$ be an open cover of B.



Then $\{U_\alpha\} \cup \{B^c\}$ is an open cover of K!

(Recall that B^c is open because B is closed.)

By compactness of K, $\exists$ finite subcover

$\{U_{\alpha_1}, \dots, U_{\alpha_n}, B^c\}$ of K.

Each $b \in B$ is not in B^c, so it must be in some U_{α_i}.

Thus $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ covers B, hence B is compact. $\square$

Cor: If $F \subset X$ is closed and X compact, then $F \cap K$ is compact.

Proof: X compact $\Rightarrow$ X closed $\Rightarrow$ $F \cap X$ closed (F is closed)

$\Rightarrow$ $F \cap X$ compact (X is compact).

$\square$