MTHSc 453

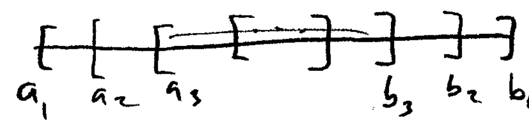
Lecture 13: The Heine-Borel Theorem

Goal: Show that in $\mathbb{R}^k$, a set is compact iff it is closed and bounded.

Theorem: Nested closed intervals in $\mathbb{R}$ have a non-empty intersection.

↳ $I_1 \supset I_2 \supset I_3 \supset \dots$, where $I_n = [a_n, b_n]$.

Proof: Let $x^* = \sup\{a_i\}$, which exists because they are bounded by b_n.



Clearly, $x^* \geq a_n$ for all a_n because it is the supremum

Also, $x^* \leq b_n$ because b_n is an upper bound for each a_m.

$\boxed{\text{If } m > n,\ a_n \leq a_m \leq b_m \leq b_n}$

Therefore, $a_n \leq x^* \leq b_n$ for all n, so $x^* \in \bigcap_{n=1}^{\infty} I_n$.

Cor: $\mathbb{R}$ is uncountable.

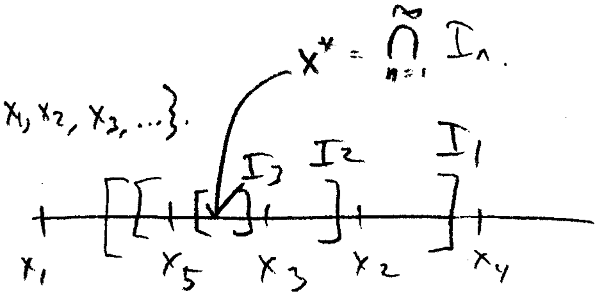
Proof: Suppose $\mathbb{R}$ were countable; $\mathbb{R} = \{x_1, x_2, x_3, \dots\}$.

Choose I_1 that misses x_1.

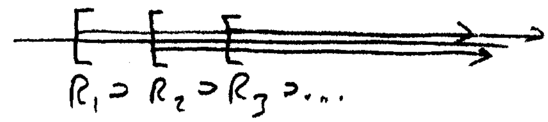
Choose $I_2 \subset I_1$ that misses x_2.

⋮

Let $x^* \in \bigcap_{n=1}^{\infty} I_n$. Then $x^* \neq x_i$ for any i! ↯ □



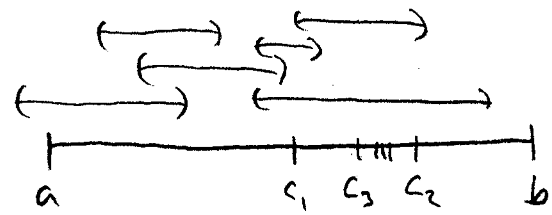
Remark: This theorem fails for closed rays.



Theorem: $[a,b]$ is compact in $\mathbb{R}$.

Proof: (by contradiction) Suppose $\exists$ open cover $\{G_\alpha\}$ that has no finite subcover. Let $I_0 = [a,b]$.

Then $\{G_\alpha\}$ covers both $[a, c_1]$ and $[c_1, b]$, where c_1 is the "center."



At least one of these subintervals has no finite subcover.

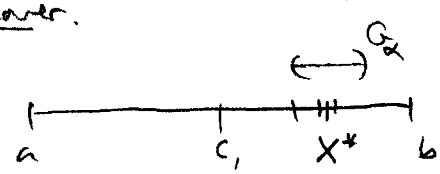
Call this interval I_1. WLOG, say $I_1 = [c_1, b]$ and let c_2 be its center. Repeat this subdivision process.

We get a sequence $I_0 \supset I_1 \supset I_2 \supset I_3 \supset \dots$

Note that (i) $|I_n| = 2^{-n} \cdot |b-a|$

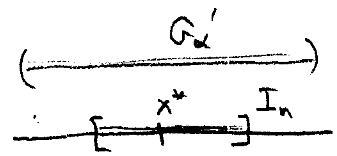
(ii) Each I_n has no finite subcover.

By Nested interval theorem, $\exists\, x^* \in \bigcap_{n=0}^{\infty} I_n$.



We know x is in some open set G_α'.

But G_α' contains some neighborhood $N_r(x) \subset G_\alpha'$.



However, for n big enough s.t. $|I_n| < 2r$, $I_n \subset G_\alpha$.

Contradicting (ii) - that I_n has no finite subcover. $\square$

Remark: This proof can be adapted for k-cells in $\mathbb{R}^k$.

$$\boxed{\text{Heine-Borel theorem: In } \mathbb{R} \text{ (or } \mathbb{R}^n\text{), [with Euclidean metric]} \quad K \text{ is compact} \iff K \text{ is closed and bounded.}}$$

Proof: ($\Rightarrow$) Already done (in all metric spaces!).

($\Leftarrow$) K bounded $\Rightarrow$ $K \subset [a,b]$ for some $a, b \in \mathbb{R}$.

Since K is closed and $[a,b]$ compact, K is compact. $\square$

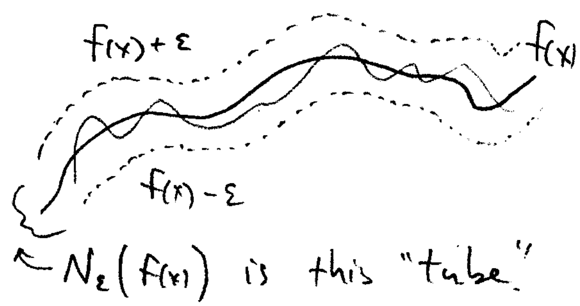
Remark: Same argument in $\mathbb{R}^n$, just with n-cells ("boxes").

Remark: The ($\Leftarrow$) direction fails for general metric spaces.

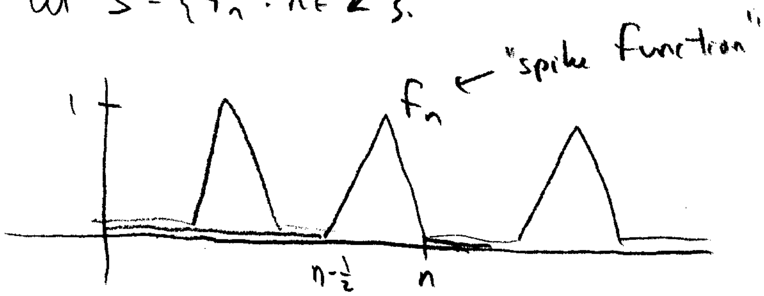
Ex: In $(\mathbb{R}, \text{disc.})$, $[a,b]$ is closed, bounded, not compact.

Ex: In $\mathscr{C}_b(\mathbb{R})$ [set of bounded cont. functions on $\mathbb{R}$].

let $d(f,g) = \sup_{x \in \mathbb{R}} |f(x) - g(x)|$



Let $S = \{f_n : n \in \mathbb{Z}\}$.



S is closed (why?) and bounded, but not compact.

Theorem: K compact $\iff$ every infinite subset of K has a limit point in K.

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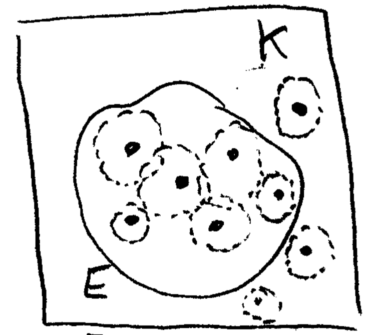

Proof: ($\Rightarrow$) If no point of K is a l.p. of E,

then • each $e \in E$ contains a nbhd U_e containing no other point of E

• each $x \notin E$ contains a nbhd V_x disjoint from $\bar{E}$.

Clearly, $\{U_e : e \in E\} \cup \{V_x : x \in K \setminus E\}$ is a cover of K that has no finite subcover, so K cannot be compact ✓

($\Leftarrow$) Much harder. On HW # 7. (through a series of exercises). $\square$

Cor (Bolzano-Weierstrass): Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$

Proof: If B is bounded, then $B \subset K$ for some compact set K.

Since $|B| = \infty$, it has a limit point. $\square$

Theorem (Cantor; "Finite intersection lemma"). Suppose $\{K_\alpha\}$ is a collection of compact sets of X. If $\bigcap_{i=1}^{n} K_{\alpha_i} \neq \emptyset$ for any finite subcollection, then $\bigcap_\alpha K_\alpha \neq \emptyset$.

Proof: Let $U_\alpha = K_\alpha^c$, open. Fix K in $\{K_\alpha\}$.

For sake of contradiction, suppose $\bigcap_\alpha K_\alpha = \emptyset$. Then $\{U_\alpha\}$ covers K.

$\Rightarrow \exists$ finite $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ covering K.

$\Rightarrow K \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n} = \emptyset$. $\lightning$ $\square$

Say that a collection $\{A_\alpha\}$ of subsets of X satisfies the <u>finite intersection property</u> (FIP) if $\bigcap_{i=1}^{n} A_{\alpha_i} \neq \emptyset$, for any finite subcollection

<u>Theorem</u> X is compact $\iff$ Any collection of closed sets $\{D_\alpha\}$ satisfies the FIP $\Rightarrow \bigcap_\alpha D_\alpha \neq \emptyset$.

<u>Proof</u>: ($\Rightarrow$) Given $\{D_\alpha\}$, closed subsets of X $\Rightarrow$ each D_α is compact.

Apply prev. theorem. ✓

($\Leftarrow$) Suppose X were not compact. Then $\exists$ open cover $\{U_\alpha\}$ with no finite subcover. Consider $\{D_\alpha\}$, where $D_\alpha = U_\alpha^c$. (closed!)

Then for any finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$, $\bigcup_{i=1}^{n} U_{\alpha_i} \neq X$,

and so $\bigcap_{i=1}^{n} D_{\alpha_n} \neq \emptyset$, but $\bigcap_\alpha D_\alpha = \emptyset$. ✓ $\square$