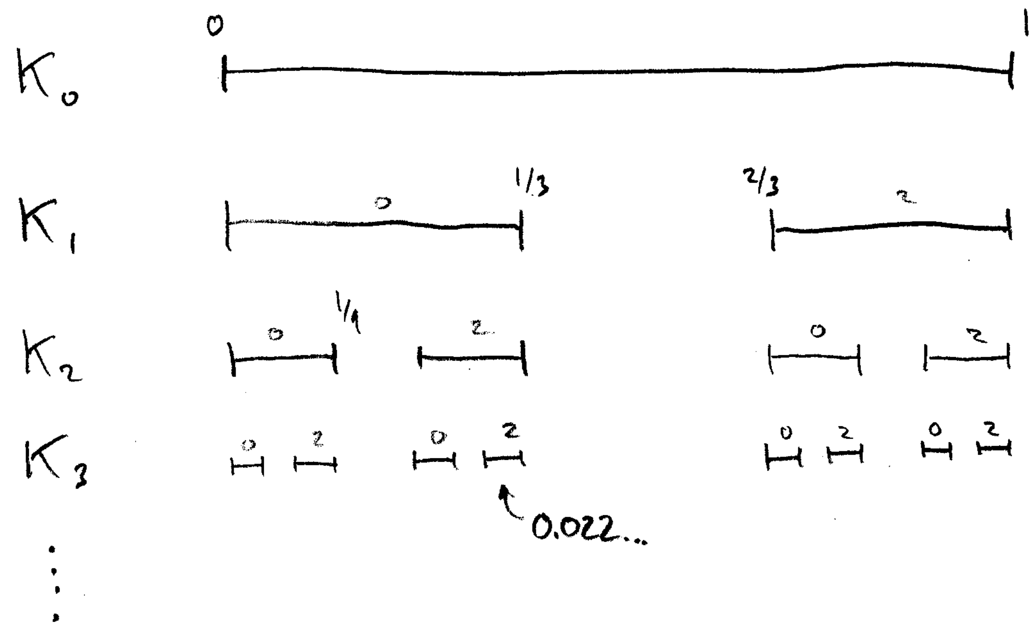
MTHSc 453

Lecture 14 Cantor sets & Connected sets.

Constructing the Cantor set:



<u>Remarks</u>: • Each K_n is closed, compact, nested.

Define the <u>Cantor set</u> as $C = \bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

- C is <u>closed</u> (intersection of closed sets)
- C is <u>perfect</u> = closed & every point is a limit point.
- C consists of real numbers whose <u>ternary</u> expansion contains only 0s and 2s.

Ternary: $\sum_{k=-\infty}^{\infty} a_k 3^k$, write $\ldots a_2 a_1 a_0 . a_1 a_2 a_3 \ldots$

Note that $\frac{1}{3} = 0.0\overline{2}$.

This shows that C is <u>uncountable</u>! (Use diagonalization argument.)

Also, the "endpoints" of K_n is <u>countable</u>, so C has points that aren't endpoints.

Endpoints are those that end in $\overline{0}$ or in $\overline{2}$.

We now see easily that every point is a limit point, and that C has no interior point.

C is <u>totally disconnected</u> (def'n later)

C has <u>measure zero</u> ($\forall \varepsilon > 0$, C can be covered by intervals of length $< \varepsilon$.)

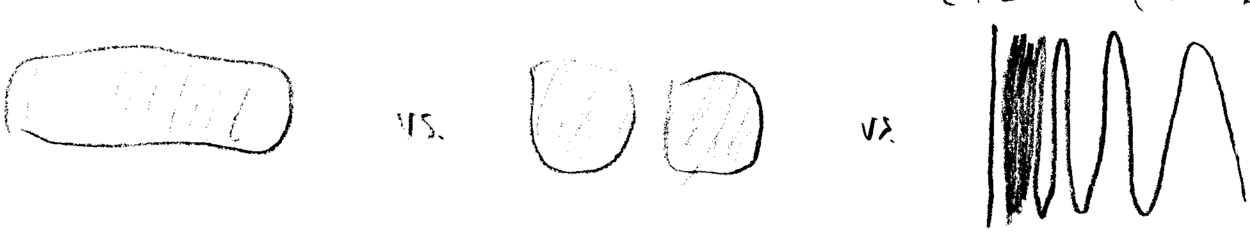
[Claim: $\mathbb{Q}$ has measure zero... how to find a cover?]

<u>Hint</u>: $\sum_{n=0}^{\infty} 2^{-n} = \frac{1}{1-1/2} = 2$.

<u>Connected sets</u>

What does it really mean for 2 sets to be connected?

$[-1,1] \cup \{\sin \frac{1}{x}\}$

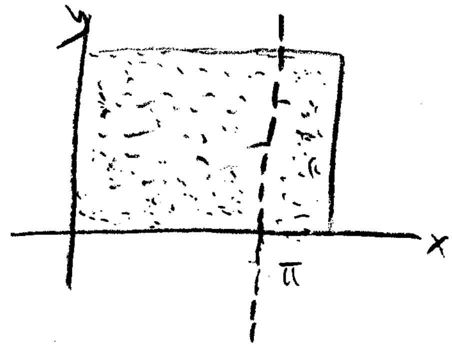


<u>Def:</u> Say A, B in X are <u>separated</u> if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty. (no point of one is a limit point of the other)

Say E is <u>connected</u> if E is <u>not</u> the union of 2 separated sets.

<u>Example</u> (in $\mathbb{R}^2$)

- $E = \{(x,y) : x, y \in \mathbb{Q}\}$ is separated



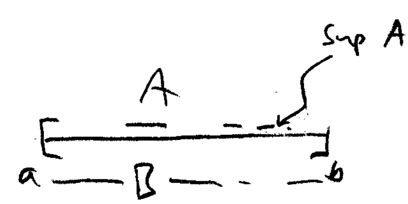
The following are equivalent:

(i) E is separated

(ii) E is the union of two nonempty <u>closed</u> sets (<u>relative</u> to E.)

(iii) E is the union of two nonempty <u>open</u> sets (<u>relative</u> to E.)

<u>Theorem:</u> $[a,b]$ is <u>connected</u>.



<u>Proof:</u> If not, then $\exists$ separation A & B, with $a \in A$.

Let $s = \sup A$. Then $s \in \bar{A}$, so $s \notin B$. $\Rightarrow s \in A$.

By def'n, $s \notin \bar{B}$. Then $\exists\ (s-\varepsilon, s+\varepsilon) \cap B = \emptyset$, hence all in A. But $s = \sup A$. ↯ $\square$